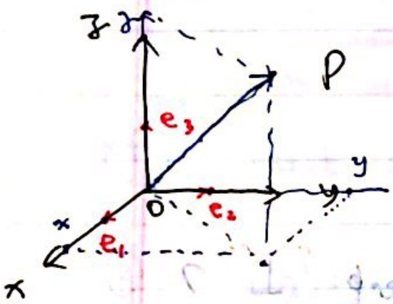
Lecture 1

§1 vector in $\mathbb{R}^3$ — $\mathbb{R}^2 \subset \mathbb{R}^3$ by $(x, y) \to (x, y, 0)$

§2 vector analysis

§3 using vector to describe geometry
Application

§1. In $\mathbb{R}^3$, we use notation $P = (x, y, z)$ for a point with respect to the coordinate system $\vec{e_1}, \vec{e_2}, \vec{e_3}$ (frame)



i.e. denote $\vec{e_1} = (1, 0, 0)$

$\vec{e_2} = (0, 1, 0)$

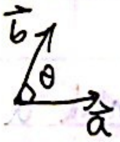
$\vec{e_3} = (0, 0, 1)$

Then $\underset{(x, y, z)}{\overset{}{P}} \overset{\text{identify}}{=\!=} \overrightarrow{OP} = x\vec{e_1} + y\vec{e_2} + z\vec{e_3}$

point — Vector "length + direction" — represent the vector in basic coordinate vector [Geometry : Projection]

Vectors can be

picture	Point	vector	represent (Calculate)
· added 	$P = (x_1, y_1, z_1)$ $Q = (x_2, y_2, z_2)$	$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$	$\overrightarrow{OR} = (x_1+x_2, y_1+y_2, z_1+z_2)$
· minus : opposite direction			$(x_1 y_1, x_2 y_2, x_3 y_3)$
· dot multiplication			



$\vec{a} \cdot \vec{b} = \underbrace{|\vec{a}| \cdot |\vec{b}| \cos\theta}_{\text{number}}$ θ = angle of $\vec{a}$ & $\vec{b}$

$\vec{a} = \overrightarrow{OP}$, $\vec{b} = \overrightarrow{OQ}$

eg: $\vec{a} \cdot \vec{b} = 0 \iff \vec{a}$ perpendicular $\vec{b}$ ($\perp$)

$\vec{e_1} \cdot \vec{e_2} = 0$, $\vec{e_i} \cdot \vec{e_j} = 0 \quad i \neq j, \; i, j = 1, 2, 3.$

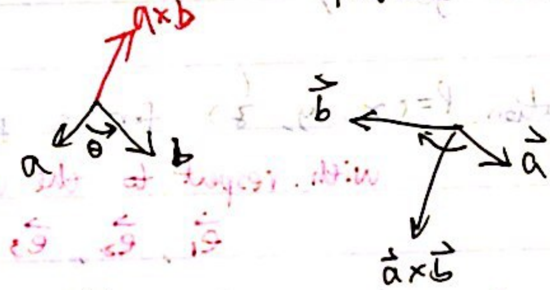
$e_i \cdot e_i = |e_i|^2 \Rightarrow |\vec{a}| = \text{length}_{\vec{a}} = \sqrt{(\vec{a} \cdot \vec{a})}$

cross product (again a vector)

$$\vec{a} \times \vec{b} = \begin{cases} \text{length} = \underbrace{|\vec{a}| \cdot |\vec{b}| \cdot \sin\theta}_{\text{geometric meaning}}, \quad \theta = \text{angle of } \vec{a} \,\&\, \vec{b} \\ \text{direction: right hand system,} \end{cases}$$

perpendicular to the plane generated by $\vec{a} \,\&\, \vec{b}$

$\neq \vec{b} \times \vec{a}$



In sum: geometric obj.

- length
- direction, in particular, how to describe perp "$\perp$" ?
- area

eg:

$\vec{c} = (-\vec{a}) + \vec{b} = \vec{b} - \vec{a}$

$\vec{c} \cdot \vec{c} = (\vec{b} - \vec{a})(\vec{b} - \vec{a})$

$= \vec{b} \cdot \vec{b} + \vec{a} \cdot \vec{a} - 2\,\vec{a} \cdot \vec{b}$

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = \text{number}$

$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

$= (-\vec{b} \times \vec{a})$

What's the meaning for this formula?

$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}| \cos\theta$ [cosine formular in triangle)

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§ Vector function

- 1 variable vector function

$V = V(t) = (x(t), y(t), z(t)) \longleftrightarrow$ curve (parameterized) : think a point moving along time t.

not the image of curve.

eg: moving from 0, with speed "1" (t, t, t)

same image $\Leftarrow$ $(2t, 2t, 2t)$ with speed "2"

· Assume $x(t), y(t), z(t)$ smooth. (i.e. differentiable)

$V'(t) = (x'(t), y'(t), z'(t))$ [just think componentwise]

Claim: ① $\left(\underbrace{\vec{a}(t) \cdot \vec{b}(t)}_{\text{number dep on } t.\text{ so it's a fun. of } t}\right)' = \underset{(\vec{a}(t))'}{\vec{a}'(t)} \cdot \vec{b}(t) + \vec{a}(t) \cdot (\vec{b}(t))'$.

pf: check each components.

② $\left(\underbrace{\vec{a}(t) \times \vec{b}(t)}_{\text{a vector dep. on } t \text{ so it is a curve}}\right)' =$

· 2 variable vector function. $\frac{d\left(\vec{a}(s,t) \cdot \vec{b}(s,t)\right)}{ds}$ = similar formula

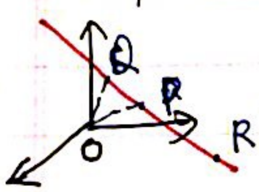
two indep. parameter (variable)

so it is a surface. (why? How to think about it)

编织 : using curve to study surface.

§3. Application: Using vector to describe geometry.

2. line equation in R^3



find the vector along the line.

||

need two point in the line

$\vec{OR} = \vec{OP} + \vec{PR} = \vec{OP} + t \cdot (\vec{PQ})$

$\vec{PQ} \parallel \vec{PR}$

1. plane equation

$$ax+by+cz+d=0$$

$$\underbrace{(a,b,c)}_{\text{normal direction}}\cdot(x,y,z) = -d$$

or you like, you can write

$$\frac{(a,b,c)}{|(a,b,c)|}\cdot(x,y,z) = \frac{d}{|(a,b,c)|}$$

$$|(a,b,c)| = \sqrt{(a,b,c)\cdot(a,b,c)}$$
$$= \sqrt{a^2+b^2+c^2}$$

MATH 321

Tutorial 2

Contents

§1. Review the math definition of curve : Parametrized smooth curve

§2. exercises P5 #2, P7 #4, P22 #2.

§1.

Def. A parametrized differentiable (smooth) curve in $\mathbb{R}^3$ is a differentiable map $\alpha: \underset{\text{open interval}}{I=(a,b)} \longrightarrow \mathbb{R}^3$

Rmk: 1. $\alpha: I \longrightarrow \mathbb{R}^3$

$t \mapsto (x(t), y(t), z(t))$, $x(t), y(t), z(t)$ are smooth functions of t.

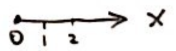
2. curve is a map. NOT the image of the map, NOT the trace of the map.

3. In this course, because we want to use calculus to study curves, we need to assume that : $\forall t \in I$, $\alpha'(t) \neq 0$; i.e, the existence of tangent line to α at t for $\forall t \in I$.

We give such curves a name : regular (parametrized differentiable) curve

4. We want to find a natural parameter : the candidate is arc length

eg: 1. A bee flys with constant speed 1 in x-direction btw time $(0,1)$ & it stops for the time $[1,2]$, then it flys away along x-direction btw time $(2,3)$ w/ speed 1



$(0,3)$

$t \to (x(t), 0, 0)$

$$x(t) = \underbrace{\begin{cases} 1\,t & t \in (0,1) \\ 1 & t \in [1,2] \\ t-1 & t \in (2,3) \end{cases}}$$

trace  $\neq$ curve, not smooth

$\lim_{t \to 1^-} x'(t) = 1 \neq \lim_{t \to 1^+} x'(t) = 0$

2. For regular (parametrized differentiable) curve

$\alpha: I \longrightarrow \mathbb{R}^3$

$t \mapsto \alpha(t)$

change variable by $S = S(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt$. think $t = t(s)$ (By inverse function Thm)

$\alpha(s) = \alpha(t(s))$

$$\dot{\alpha} = \alpha'(t)\frac{dt}{ds}$$

Reserve the notation " $\cdot$ " for arc length parameter, i.e $\dot{\alpha} := \frac{d\alpha(s)}{ds}$

t is the arc length parameter $\Longleftrightarrow$ $|\alpha'(t)| = 1$.

Rmk: ① " parametrized regular curve / $\sim$ " in this class. we pick up a representative "arc length" parametrized curve

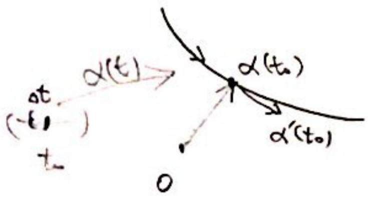
change of parameter

③ " param. regul curve/$\sim$ & $\frac{\partial t}{\partial s} > 0$ " oriented

§2. ② Study local property : curvature & torsion

PS #2 : Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

$\alpha : (a, b) \to \mathbb{R}^3$



$$\alpha(t) \cdot \alpha(t) = |\alpha(t)|^2 \neq 0 \quad \forall\, t \in (a, b)$$

now $t_0 \in (a, b)$ s.t.

$$|\alpha(t_0)| = \min_{t\in(a,b)} |\alpha(t)|$$

i.e.

$$|\alpha(t_0)|^2 = \min_{t\in(a,b)} |\alpha(t)|^2$$

Def.

$$g(t) : (a, b) \longrightarrow \mathbb{R}$$
$$t \mapsto |\alpha(t)|^2$$

then $g(t)$ is a smooth function, $g(t) > 0$

$$g(t_0) = \min_{t\in(a,b)} g(t)$$

$$\boxed{g(t) \text{ obtains its minimal (at } t_0) \Rightarrow g'(t_0) = 0}$$

$$\lim_{\Delta t \to 0} \frac{\overbrace{g(t_0+\Delta t) - g(t_0)}^{\geq 0}}{\Delta t}$$

$\Delta t \to 0^+$ ≥ 0
$\Delta t \to 0^-$ ≤ 0 $\Big\} = 0$

$$g'(t_0) = 2\alpha'(t_0) \cdot \alpha(t_0)$$

$$\Rightarrow \alpha'(t_0) \cdot \alpha(t_0) = 0 \qquad \alpha'(t_0) \perp \alpha(t_0).$$

P#4. Let $\alpha : (0, \pi) \to R^2$

$$t \mapsto (\sin t,\ \cos t + \log \tan \tfrac{t}{2})$$

where t is the angle that the y axis makes with the vector $\alpha'(t)$.

The trace of α is called the tractrix.

Show that

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$

b. The length of the segment of the tangent of the tractrix between the point of tangency & the y axis is always 1.

pf. a. $\frac{t}{2} \in (0, \frac{\pi}{2})$ $\quad \tan \frac{t}{2} \in (0, +\infty)$

so $\log \tan \frac{t}{2}$ has definition for $t \in (0, \pi)$.

Since all those basic functions (sin, cos, log, tan) are differentiable functions in its definition domain, by the composition law $\cos t + \log \tan \frac{t}{2}$ is differentiable function $\Rightarrow$

α is a diff. para. curve.

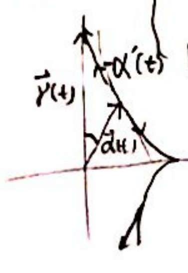
$$\alpha'(t) = \left(\cos t,\ -\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \frac{1}{\cos^2(\frac{t}{2})} \cdot \frac{1}{2}\right)$$

$$= \left(\cos t, -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = \left(\cos t, -\sin t + \frac{1}{\sin t}\right) = \left(\cos t, \frac{\cos^2 t}{\sin t}\right)$$

$\alpha'(\frac{\pi}{2}) = (0, 0)$ $\quad$ & $\quad \alpha'(t) \neq 0 \quad t \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$

b. $\alpha(t) + \beta(t) = \gamma(t)$

$$\beta(t) = \begin{cases} |\beta(t)| \left(-\dfrac{\alpha'(t)}{|\alpha'(t)|}\right) & t \in (0, \frac{\pi}{2}) \\ |\beta(t)| \dfrac{\alpha'(t)}{|\alpha'(t)|} & t \in (\frac{\pi}{2}, \pi) \end{cases}$$



$$|\alpha'(t)| = \sqrt{\cos^2 t + \cos^4 t / \sin^2 t} = |\cos t| \frac{1}{|\sin t|}$$

$$= \begin{cases} \dfrac{\cos t}{\sin t} & t \in (0, \frac{\pi}{2}) \\ -\dfrac{\cos t}{\sin t} & t \in [\frac{\pi}{2}, \pi) \end{cases}$$

$\gamma(t) = (0, *)$

$$\sin t - |\beta(t)| \frac{\cos t\,|\sin(t)|}{|\cos t|} = 0 \quad t \in (0, \tfrac{\pi}{2})$$

$$\sin t + |\beta(t)| \frac{\cos t}{(-\cos t/\sin t)} = 0 \quad t \in [\tfrac{\pi}{2}, \pi)$$

$\Rightarrow |\beta(t)| = 1$ $\quad \square$

Pr #2 Show that the torsion τ of α is given by

$$\tau(s) = - \frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}$$

Notation Remark
if s is arc length
$\alpha'(s) = \dot{\alpha}(s)$

Recall $\alpha : I \to R^3$ s : arc length
$s \mapsto \alpha(s)$

$\alpha \to \dot{\alpha} = t$ tangent

$t' \cdot t = 0$

$\frac{t'}{|t'|} = n$

$t' = k(s)\, n$

$b := t \wedge n$

$t(s) = \alpha'(s)$

$t'(s) = \alpha''(s)$ $|t'(s)| = |\alpha''(s)| := k(s)$

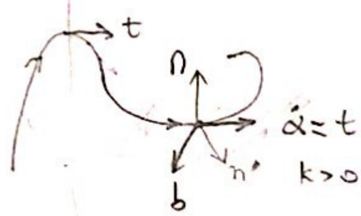
$= k(s)\, n$

$b(s) := t(s) \wedge n(s)$

$b'(s) = t'(s) \wedge n(s) + t(s) \wedge n'(s)$

$= t(s) \wedge n'(s) \Rightarrow b'(s) \,//\, n(s)$

draw picture



Def $\tau(s)$ s.t. $b'(s) = \tau(s)\, n(s)$

Pf: change everything back by Frenet formulas

$t' = kn$

$n' = -kt - \tau b$

$b' = \tau n$

$\alpha'(s) = t(s)$ $\alpha''(s) = k(s)\, n(s)$

$\alpha'''(s) = (k(s)\, n(s))' = k'(s)\, n(s) + k(s)\, n'(s)$

$= k'(s)\, n(s) + k(s)(-k(s) t(s) - \tau(s) b(s))$

$= k'(s)\, n(s) - k^2(s)\, t(s) - k(s)\tau(s)\, b(s)$

$\alpha'(s) \wedge \alpha''(s) = k(s)\, t(s) \wedge n(s) = k(s)\, b(s)$

$(k(s)\, b(s)) \cdot (\alpha'''(s)) = -k^2(s)\, \tau(s)\, b(s) \cdot b(s) = -k^2(s)\, \tau(s)$

$$\Rightarrow \quad \tau(s) = - \frac{(-k^2(s)\, \tau(s))}{(k(s))^2}$$

RM 4620. (L# 31-32) ①
Mod, 18 – 18:50 (课内发作业 也收作业
下课 7:00 (Box 51) 收作业

MATH321. Tutorial 3

Contents P_{25} # 13 P_{26} # 15.

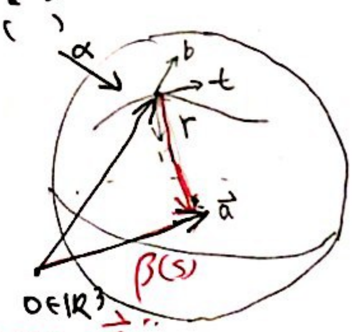
#13 $\alpha: I=(a,b) \longrightarrow \mathbb{R}^3$

Assume that $\tau(s) \neq 0$ & $k(s) \neq 0$ for all $s \in I$

Show that a necessary & sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$\boxed{R^2 + (R')^2 T^2 = \text{constant}} \quad ☆$$

where $R = 1/k(s)$, $T = 1/\tau(s)$, $R' = \frac{dR}{ds}$, s arc length.



Observe: $\frac{d}{ds}(☆)$: $2RR' + 2R'R''T^2 + (R')^2\, 2TT' = 0$

where $T' = \frac{dT}{ds}$

$R' = \left(\frac{1}{k(s)}\right)' = -\frac{k'(s)}{k^2(s)} \neq 0$ by assumption.

(Recall $k(s) > 0$ in the def. of τ. P_{18})

So (☆) $\Leftrightarrow$ $R + R''T^2 + R'TT' = 0$.

$\Rightarrow$ If $\alpha(s)$ lies on sphere, assume center is $\vec{a}$, $r := \alpha(s) - \vec{a}$

then $r \cdot r = \text{const.}$ $\quad r' = \alpha' = t$

$\Downarrow$

$r \cdot r' = 0$ i.e $r \cdot t = 0$

So assume $r = y(s)\,n + z(s)\,b = -Rn + R'T b$ $\quad \begin{cases} t' = kn \\ n' = -kt - \tau b \\ b' = \tau n \end{cases}$

$t = r' = y'n + yn' + z'b + zb'$ ← apply Frenet Frame

$= (-y \cdot k)\,t + (y' + z\tau)\,n + (z' - y\tau)\,b$

So $\begin{cases} y' + z\tau = 0 & ① \\ z' - y\tau = 0 & ② \end{cases}$

$y \cdot k = -1 \quad y = -\frac{1}{k} = -R$

On the other hand, $\quad z = \frac{-y'}{\tau} = R'T$

$(r \cdot t)' = 0 \Rightarrow r' \cdot t + r \cdot kn = 0$, i.e, $1 + k(r \cdot n) = 0$

t $\quad 1 + ky = 0$

$y = -\frac{1}{k} = -R$, put in ①

$(-R)' + z\frac{1}{T} = 0 \Rightarrow z = R'T \underset{\text{put into ②}}{\Rightarrow} (R'T)' + R\frac{1}{T} = 0$

i.e $\boxed{R''T^2 + R'T'T + R = 0}$

$\Leftarrow$) (☆) .

"$\Leftarrow$" (need to find a const. vect. $\vec{a}$
s.t $r = (\alpha(s) - \vec{a})$ satisfying $r \cdot r = \text{const.}$
or $r' \cdot r = 0$
i.e $t \cdot r = 0$)

with the help of "$\Rightarrow$". Let us try to compute

$$\beta(s) := \alpha(s) - \vec{r} = \alpha(s) - (-Rn + R'Tb)$$

$$= \alpha(s) + Rn - R'Tb$$

Claim: $\beta(s)$ is a constant vector.

Pf: $$\beta'(s) = \alpha'(s) + Rn' + R'n - (R'T)'b - (R'T)b'$$

$$= t + R(-kt - \tau b) + R'n - (R'T)'b - (R'T)\tau n$$

$$\overset{\substack{Rk=1 \\ T\tau=1}}{=} \cancel{t - t} - R\tau b + R'n - (R'T)'b - R'n$$

$$= -(R\tau + (R'T)')\, b$$

$$= -\frac{R + (R'T)'T}{T}\, b$$

$$= -\frac{R + R''T^2 + R'T'T}{T}\, b = 0 \quad \text{by (☆)}$$

$\Rightarrow \beta(s) = a$ constant vector.

Then $$|\alpha(s) - a|^2 = |\vec{r}|^2 = R^2 + (R'T)^2 = \text{const.}$$

$\alpha(s)$ lies in the sphere with center a, radius $\sqrt{R^2 + (R'T)^2}$

□

#15 Show that the knowledge of the vector function $b = b(s)$ (binormal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $k(s)$ and the absolute value of the torsion $\tau(s)$ of α.

Review: • We know $b = b(s)$ as binormal vector of a curve α,

But we don't know the curve $\alpha = \alpha(s)$ itself.

可以先讲这些再讲两道题目. 也可以省略这部分而直接用

• Tool: Frenet frame

If know $\alpha = \alpha(s)$.

then $|\alpha'(s)| = 1$ $\quad$ s as arclength,

$\alpha'(s) \cdot \alpha'(s) = 1 \Rightarrow \alpha'(s) \cdot \alpha''(s) = 0$ $\qquad t(s) := \alpha'(s)$

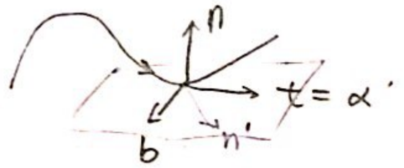
$k(s) := |\alpha''(s)|$, $\quad n(s) := \dfrac{\alpha''(s)}{|\alpha''(s)|}$ $\quad$ normalize $\alpha''(s)$

(So $t' = \alpha'' = k(s)\, n(s)$) ①

$b := t(s) \wedge n(s)$ ③

$b' = t'(s) \wedge n(s) + t(s) \wedge n'(s)$

$= k(s) \underbrace{n(s) \wedge n(s)}_{=0} + t(s) \wedge n'(s)$



Question, where is $n'(s)$?

answer: $n' \perp n$ (Recall $|n| = 1$, $n \cdot n = 1 \Rightarrow n' \cdot n = 0$)

n' on the plane spanned by b & t

$\Rightarrow$ Therefore $\boxed{t(s) \wedge n'(s) \,/\!/\, n \quad ☆}$

So we can write $b' = (*)\, n$

actually, we define such $(*)$ to be $\tau(s)$

$b' = \tau(s) n(s)$ ②

Then $n'(s) = (b \wedge t)' = b' \wedge t + b \wedge t' = \underbrace{\tau(s)}_{②} \underbrace{n(s) \wedge t(s)}_{-b} + b \wedge (\underbrace{k(s)\, n(s)}_{①})$

$= -\tau(s)\, b - k(s)\, t(s)$ ④

in sum $\Longrightarrow$
$$\begin{cases} t' = k n & ① \text{ (definition of } k\text{)} \\ n' = -kt - \tau b & ④ \text{ (computation of ①, ② \& ③)} \\ b' = \tau n & ② \text{ (observation ☆ \& definition of } \tau\text{)} \end{cases}$$

Now.

$$b' = \tau n \qquad \text{so} \qquad |\tau| = |b'|$$

We know $|\tau|$ from knowledge of b.

$$\begin{aligned} b'' &= \tau' n + \tau n' \\ &= \tau' n + \tau(-k t - \tau b) \\ &= \tau' n - k\tau t - \tau^2 b \end{aligned}$$

// Recall, we want k

$$b'' + \tau^2 b = \tau' n - k\tau t$$

$$\begin{aligned} (b'' + \tau^2 b)' &= \tau'' n + \tau' n' - (k\tau)' t - (k\tau) t' \\ &= \tau'' n + \tau'(-kt - \tau b) - (k\tau)' t - (k\tau) k n \\ &= (\tau'' - k^2\tau) n + (-\tau' k - (k\tau)') t - \tau'\tau b \end{aligned}$$

we know this term

$$(b'' + \tau^2 b)' \cdot b' = (\tau\tau'' - k^2\tau^2)\, n \cdot n = \tau\tau'' - k^2\tau^2$$

We know $|\tau|$, $\Rightarrow \tau^2 = b' \cdot b'$

$$(\tau^2)' = 2\tau\tau' \Rightarrow ((\tau^2)')^2 = 4\tau^2(\tau')^2$$

$$\begin{aligned} (\tau^2)'' &= 2\tau'\tau' + 2\tau\tau'' \\ &= 2(\tau')^2 + 2\tau\tau'' \end{aligned}$$

$$\begin{aligned} \Rightarrow \tau\tau'' &= (\tau^2)'' - 2(\tau')^2 \\ &= (\tau^2)'' - \frac{2((\tau^2)')^2}{4\tau^2} \end{aligned}$$

$$\begin{aligned} k^2\tau^2 &= \tau\tau'' - (b'' + \tau^2 b)' \cdot b' \\ &= (\tau^2)'' - \frac{((\tau^2)')^2}{2\tau^2} - (b'' + \tau^2 b)' \cdot b' \end{aligned}$$

where $\tau^2 = b' \cdot b'$

$\Rightarrow k^2$

$\Rightarrow k \quad (k > 0)$

$\square$

-Tutorial 4 MATH 321 2010 Sept 27

Contents. P24 #10, P26 #18

#10 Consider the map

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}) & t>0 \\ (t, e^{-1/t^2}, 0) & t<0 \\ (0, 0, 0) & t=0 \end{cases}$$

a. Prove that α is a differentiable curve

b. Prove that α is regular for all t
and that the curvature $k(t) \neq 0$ for $t \neq 0$, $t \neq \pm\sqrt{2/3}$
$k(0) = 0$

c. Show that the limit of the osculating planes as $t \to 0$, $t>0$ is the plane $y=0$ but that the limit of the osculating planes as $t \to 0$, $t<0$ is the plane $z=0$ (this implies that the normal vector is discontinuous at $t=0$ & shows why we excluded points where $k=0$).

d. Show that τ can be defined so that $\tau \equiv 0$, even though α is not a plane curve. [See Prop 1.11 & 1.12 of prof. Li's notes, $\alpha''(s) \neq \vec{0}$ $\forall s$ i.e. $k(s) > 0$ $\forall s$.]

$\boxed{\text{So with the assumption } k>0 \text{ we have } \tau = 0 \Leftrightarrow \alpha \text{ is plane curve}}$

Observation

1. $y = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ draw the graph of this function

- $y > 0$
- even function $y(x) = y(-x)$
- $\lim\limits_{\Delta x \to 0} \dfrac{y(0+\Delta x) - y(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{e^{-1/(\Delta x)^2} - 0}{\Delta x}$ " $\dfrac{0}{0}$ " indeterminate form

(L'Hôpital's rule)

$\Delta x = \frac{1}{s}$

$$\lim_{s\to\infty} \frac{s}{e^{s^2}} = \lim_{s\to\infty} \frac{(s)'}{(e^{s^2})'} = \lim_{s\to\infty} \frac{1}{e^{s^2}\cdot 2s} = 0$$

So $\lim\limits_{\Delta x \to 0} \dfrac{y(0+\Delta x) - y(0)}{\Delta x} = 0 = y(0)$

y is continuous at 0.

Review L'Hôpital's Rule

" $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ "

(when $\frac{f(x)}{g(x)}$ indeterminate form near the point a)

$\frac{0}{0}$, $\frac{\infty}{\infty}$

Claim. y is differentiable at $x=0$, & for any positive integer k the k-th derivative of y at 0 is 0, i.e. $y^{(k)}(0)=0$.

For $x\neq 0$ $\quad y'(x) = (e^{-1/x^2})' = e^{-1/x^2}\cdot\left(-\frac{1}{x^2}\right)' = e^{-1/x^2}\cdot\frac{2}{x^3}$

Define $y'(0)=0$. Now check $1^\circ\ \lim_{\Delta x\to 0} y'(\Delta x)=0$ [By L'Hôpital's Rule

$\Delta x=\frac{1}{s}$ $\quad \lim_{s\to\infty}\frac{3s^3}{e^{s^2}} = \lim_{s\to\infty}\frac{9s^2}{2se^{s^2}}$

$= \lim_{s\to\infty}\frac{18s}{2e^{s^2}+(2s)^2e^{s^2}} = \lim_{s\to\infty}\frac{18}{(\quad)'}$

$=0.$

& 2° $\quad \underbrace{\lim_{\Delta x\to 0}\frac{y'(\Delta x)-y'(0)}{\Delta x}}_{y''(0)} = 0$ (By the same method).

In general, By induction, we assume $y^{(k)}$ is of the form

$$y^{(k)}(x)=\begin{cases} e^{-1/x^2}\cdot\dfrac{P_k(x)}{Q_k(x)} & x\neq 0, \text{ where } \lim_{x\to 0}\dfrac{P(x)}{Q(x)}=\infty \\ 0 & x=0\end{cases}$$

$\cdot\ P(x), Q(x)$ polynomial

- $\lim_{\Delta x\to 0} y^{(k)}(\Delta x)=0$ $\quad$ i.e. $y^{(k)}$ is continuous at 0.
- $\lim_{\Delta x\to 0}\frac{y^{(k)}(\Delta x)-y^{(k)}(0)}{\Delta x} \overset{\Delta x=s^{-1}}{=} \lim_{s\to\infty}\frac{\tilde P(s)}{e^{s^2}\cdot\tilde Q(s)}$ $\quad$ $\tilde P(s)$, $\tilde Q(s)$ poly of s

$= \lim_{s\to\infty}\frac{(\tilde P(s))'}{(e^{s^2}\tilde Q(s))'}$ ... after $\deg \tilde P(s)$-th derivative L'Hôpital Rule

$=0.$

$\lim_{\Delta x\to 0} y^{(k+1)}(\Delta x)=0$

$\Rightarrow$ $y^{(k+1)}$ is of the form $\quad y^{(k+1)}(x)=\begin{cases} e^{-1/x^2}\dfrac{P_{k+1}(x)}{Q_{k+1}(x)} \\ 0\end{cases}$ $\quad$ Same polynomial.

Therefore

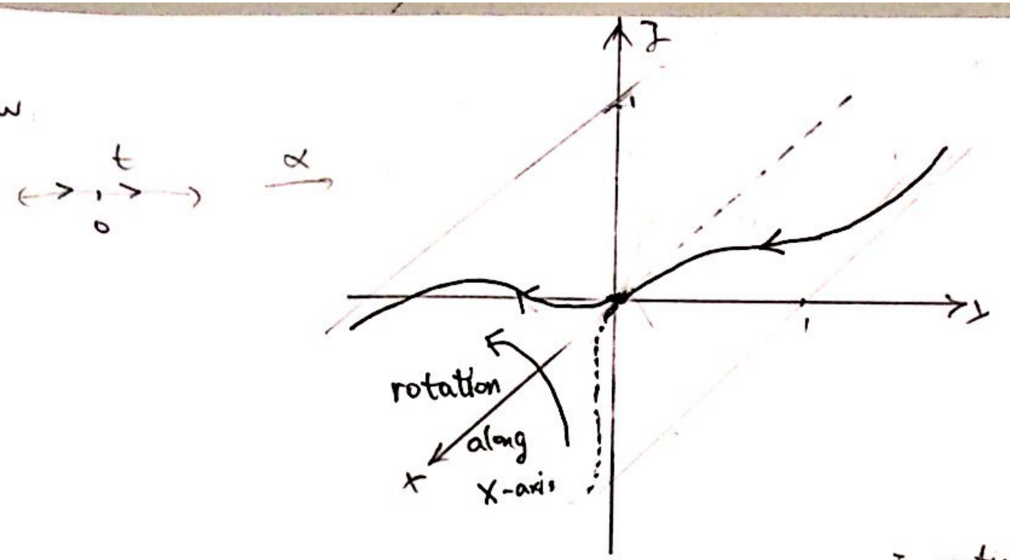
$y^{(k)}(0)=0$ $\quad\forall k$ positive integer

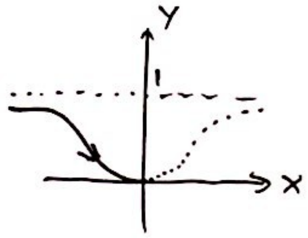
$y=y(x)$ is "flat" at $x=0$, i.e. $y=y(x)$ as flat as a line at the point $x=0$

$y=y(x)$ & $y=0$, these two functions are the same at $x=0$.

So we can rotate the graph of $y=y(x)$ along "the line" at $x=0$ (x-axis)

Now





3

$$\alpha'(t) = \begin{cases} \left(1,\ 0,\ e^{-1/t^2}\dfrac{2}{t^3}\right) & t>0 \\ \left(1,\ e^{-1/t^2}\dfrac{2}{t^3},\ 0\right) & t<0 \\ (1,\ 0,\ 0) & t=0 \end{cases}$$

In particular

$$\alpha'(0) = \lim_{\Delta t \to 0} \frac{\alpha(\Delta t) - \alpha(0)}{\Delta t} = (1,0,0)$$

so $\alpha'(t) \neq \vec{0}\ \forall t$

$\Rightarrow \alpha$ is regular.

(a)

$$\alpha''(t) = \begin{cases} \left(0,\ 0,\ e^{-1/t^2}\left(\dfrac{4}{t^6}-\dfrac{6}{t^4}\right)\right) & t>0 \\ \left(0,\ e^{-1/t^2}\left(\dfrac{4}{t^6}-\dfrac{6}{t^4}\right),\ 0\right) & t<0 \\ (0,\ 0,\ 0) \end{cases}$$

$$\alpha''(0) = \lim_{\Delta t \to 0} \frac{\alpha'(\Delta t) - \alpha'(0)}{\Delta t} = (0,0,0)$$

$$\alpha'(t) \wedge \alpha''(t) = \begin{cases} \left(0,\ -e^{-1/t^2}\left(\dfrac{4}{t^6}-\dfrac{6}{t^4}\right),\ 0\right) & t>0 \\ \left(0,\ 0,\ e^{-1/t^2}\left(\dfrac{4}{t^6}-\dfrac{6}{t^4}\right)\right) & t<0 \\ (0,\ 0,\ 0) & t=0 \end{cases}$$

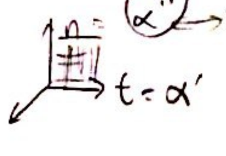
$$\begin{vmatrix} i & j & k \\ 1 & 0 & e^{-1/t^2}\cdot\frac{1}{t^3} \\ 0 & 0 & e^{-1/t^2}\left(\frac{4}{t^6}-\frac{6}{t^4}\right) \end{vmatrix} = \left(0,\ -e^{-1/t^2}\left(\frac{4}{t^6}-\frac{6}{t^4}\right),\ 0\right)$$

(b) $$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3} = \frac{e^{-1/t^2}\left|\frac{4}{t^6}-\frac{6}{t^4}\right|}{\sqrt{1+e^{-2/t^2}\frac{1}{t^6}}}$$

$k(t) \neq 0$ for $t \neq 0$ & $t \neq \pm\sqrt{\frac{2}{3}}$

$k(0) = 0$. (You can also reparametrize this curve by arclength parameter. $\alpha''(s)\big|_{s=0} = (0,0,0)$
$k(0) = |\alpha''(0)| = 0$)

(c): osculating plane



$t = (1,0,0)$

$\alpha'' \to n = \begin{cases} (0,0,1) & t>0 \\ (0,1,0) & t<0 \\ (0,0,0) & t=0 \end{cases}$

$$b = \begin{cases} (0,-1,0) & t>0 & y=0 \\ (0,0,1) & t<0 & z=0 \\ (0,0,0) & t=0 \end{cases}$$

(d) chain rule

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'')\cdot\alpha'''}{|\alpha' \wedge \alpha''|^2}$$

$$\alpha'''(t) = \begin{cases} (0,\ 0,\ *) \\ (0,\ *,\ 0) \\ (0,\ 0,\ 0) \end{cases}$$

$\alpha'''(t) \parallel \alpha''(t)$

$\tau \equiv 0$

□

18 $\alpha : I \to R^3$ parametrized regular curve, $k(t) \neq 0$, $\tau(t) \neq 0$ $t \in I$ ④

The curve α is called Bertrand curve if

$\exists$ curve $\bar{\alpha} : I \to R^3$ s.t.

the normal lines of α & $\bar{\alpha}$ at $t \in I$ are equal. $\Rightarrow$ $\boxed{n \parallel \bar{n}}$

In this case, $\bar{\alpha}$ is called a Bertrand mate of α, and we can write

$$\bar{\alpha}(t) = \alpha(t) + r(t) n(t)$$

number

Prove that

a. r is constant. $r(t) = r$.

b. α is a Bertrand curve iff $\exists$ linear relation $A\,k(t) + B\,\tau(t) = 1$, $t \in I$ where A, B nonzero constants

c. If α has more than one Bertrand mate, it has infinitely many Bertrand mates. This case occurs iff α is a circular helix.

Pf a). We parametrize α by arclength s, (we may assume t is the arclength)
$\bar{\alpha}$ 〃 $\bar{s}$

$$\alpha = \alpha(s)$$

$$\bar{\alpha} = \alpha(s) + r(s) n(s)$$

$$\frac{d\bar{\alpha}}{ds} = \alpha'(s) + r'(s) n + r n' = t + r' n + r(-kt - \tau b)$$
$$= (1 - kr) t + r' n + (-\tau r) b$$

$\frac{d\bar{\alpha}}{ds}$ is tangent to $\bar{\alpha}$ $\Rightarrow$ $\frac{d\bar{\alpha}}{ds} \cdot \bar{n} = 0$ $\Rightarrow$ $\frac{d\bar{\alpha}}{ds} \cdot n = 0$

$\Rightarrow$ $r' = 0$ $\Rightarrow$ $r = \text{constant}$

b).

⑤

" $\Rightarrow$ " $\bar{t}$: unit tangent vector of $\bar{\alpha}$, i.e., $\bar{t} = \frac{d\bar{\alpha}}{d\bar{s}} = \frac{d\bar{\alpha}}{ds}\frac{ds}{d\bar{s}}$

$$\frac{d}{ds}(t\cdot\bar{t}) = t\cdot\frac{d\bar{t}}{ds} + \frac{dt}{ds}\cdot\bar{t}$$

$$= t\cdot\frac{d\bar{t}}{d\bar{s}}\left(\frac{d\bar{s}}{ds}\right) + \frac{dt}{ds}\cdot\bar{t}$$

$$= \underbrace{t\cdot\bar{n}}_{0}\frac{d\bar{s}}{ds} + \underbrace{n\cdot\bar{t}}_{0}$$

$n \parallel \bar{n}$

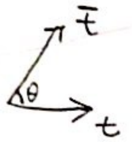
$$= 0 + 0 = 0$$

$$= \left(\frac{d\alpha}{ds} + r\frac{dn}{ds}\right)\frac{ds}{d\bar{s}}$$

$$= (t + r(-kt - \tau b))\frac{ds}{d\bar{s}}$$

$$= ((1-rk)t - r\tau b)\frac{ds}{d\bar{s}}$$

$\Rightarrow$ $t\cdot\bar{t} = \cos\theta$



$|t| = |\bar{t}| = 1$

$\parallel$

$$t\cdot(1-rk)t\frac{ds}{d\bar{s}} = (1-rk)\frac{ds}{d\bar{s}}$$

$$|t\wedge\bar{t}| = |\sin\theta|$$

$\parallel$

$$\left|r\tau\frac{ds}{d\bar{s}}\right|$$

$$\frac{1-rk}{r\tau} = c \quad \text{Constant} \quad (*)$$

Rewrite this relation, we get

$$Ak + B\tau = 1 \quad (\star)$$

$A = r$, $B = (cr)$ constants.

" $\Leftarrow$ "

Conversely, if we have relation $(\star)$, i.e. $(*)$,

Define
$$\bar{\alpha}(s) = \alpha(s) + An(s) = \alpha(s) + rn(s)$$

Try to prove the normal lines of α & $\bar{\alpha}$ at $s\in I$ are equal

$$\frac{d\bar{\alpha}(s)}{ds} = \frac{d(\alpha(s) + rn(s))}{ds} = (1-rk)t - r\tau b$$

$$\overset{(*)}{=} B\tau t - r\tau b$$

$$= \tau(Bt - rb)$$

normalize this vector : $\bar{t} = \frac{Bt - rb}{\sqrt{B^2+r^2}}$

$\frac{dt}{ds} = kn$, $\frac{db}{ds} = \tau n$

$$\frac{d\bar{t}}{ds} = \frac{Bkn - r\tau n}{\sqrt{B^2+r^2}} = \frac{(Bk - r\tau)}{\sqrt{B^2+r^2}}n$$

$$\bar{n} = \frac{d\bar{t}}{d\bar{s}} = \frac{d\bar{t}}{ds}\frac{ds}{d\bar{s}}$$

vector, scalar

= normalize of vector $\frac{d\bar{t}}{ds}$

MATH 321. Tutorial 5.

Contents : P88~89, #4. #6 #7.

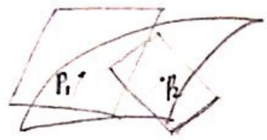
#4. Show that the tangent planes of a surface given by

$z = x f(y/x)$, $x \neq 0$, where f is a differentiable function, all pass through the origin $(0,0,0)$.

Analysis :

1. The equation of the tangent plane of the surface passing through the point $P = (x, y, z)$ is

$$\vec{N} \cdot ((X, Y, Z) - \vec{OP}) = 0$$



2. For $z = x f(y/x)$

$$\vec{N} = ((x f(y/x))_x, (x f(y/x))_y, -1)$$

$'o$

Proof : $$\vec{N} = \left(f(\tfrac{y}{x}) + x f'(\tfrac{y}{x}) \tfrac{-y}{x^2},\ x f'(\tfrac{y}{x}) \cdot \tfrac{1}{x},\ -1 \right) \qquad x \neq 0$$

$$= \left(f(\tfrac{y}{x}) - f'(\tfrac{y}{x}) \tfrac{y}{x},\ f'(\tfrac{y}{x}),\ -1 \right)$$

Then the tangent plane is

$$\circledast \quad \vec{N} \cdot (X - x, Y - y, Z - z) = 0$$, where X, Y, Z are parameters of the plane.

To check that this plane passes through $(0,0,0)$, we only need to put $X=0$ $Y=0$ $Z=0$ into $\circledast$ and check they satisfy the equation, ie.

$$\vec{N} \cdot (-x, -y, -z) \overset{?}{=} 0$$ check

By Computation :

$$\vec{N} \cdot (-x, -y, -z) = \left(f(\tfrac{y}{x}) - f'(\tfrac{y}{x}) \tfrac{y}{x} \right)(-x) + f'(\tfrac{y}{x})(-y) + z$$

$$= -x f(\tfrac{y}{x}) + z = 0.$$

$\square$

②

#6. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with everywhere nonzero curvature. Consider the tangent surface of α.

$$X(t, v) = \alpha(t) + v\alpha'(t) \qquad t \in I,\ v \neq 0.$$

Show that the tangent planes along the curve $X(\underset{\substack{\uparrow \\ \text{constant}}}{t_0}, v)$ are all equal.

pf.

Step 1. Write down the tangent plane equation of the surface $X(t, v)$ at the point (t_0, v)

Step 2. Check the equation is independent of v.

Step 1): $\vec{X}_t = \alpha'(t) + v\alpha''(t)$ $\qquad [K_{\alpha(t)} > 0 \Rightarrow \alpha''(t) \neq 0.]$

$\vec{X}_v = \alpha'(t)$

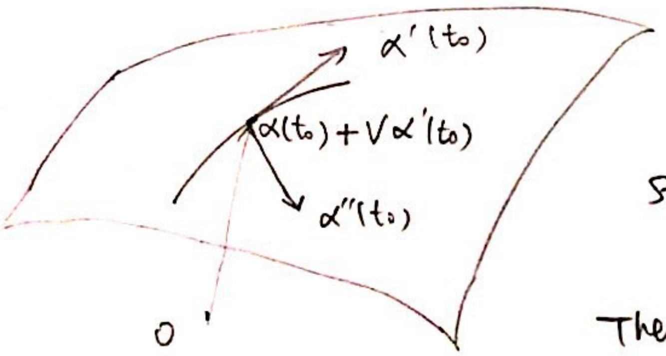
$$\vec{X}_t \times \vec{X}_v = (\alpha'(t) + v\alpha''(t)) \times \alpha'(t)$$

$$= v\,\alpha''(t) \times \alpha'(t).$$ normal direction of the tangent plane

$\Rightarrow$ Tangent plane equation at the point $X(t_0, v)$:

$$(v\,\alpha''(t_0) \times \alpha'(t_0)) \cdot ((X, Y, Z) - X(t_0, v)) = 0$$

i.e.
$$v\,\alpha''(t_0) \times \alpha'(t_0) \cdot (X, Y, Z)$$
$$- v\,\alpha''(t_0) \times \alpha'(t_0) \cdot (\alpha(t_0) + v\alpha'(t_0)) = 0$$



i.e

$$\boxed{\alpha''(t_0) \times \alpha'(t_0) \cdot ((X, Y, Z) - \alpha(t_0)) = 0}$$

Step 2): It is obvious that the above equ. is independent of v.

Therefore we get the conclusion

□

#7. Let $f: S \to \mathbb{R}$ be given by $f(p) = |p - p_0|^2$ where $p \in S$

$p_0 \in \mathbb{R}^3$ fixed. Show that $df_p(w) = 2w \cdot (p - p_0)$ $w \in T_p(S)$

Analysis:

1. what is $T_p(S)$

2. how to define df_p

check the definition of $d\varphi_p$ for $\varphi: V \overset{\text{open}}{\subset} S_1 \to S_2$ (Page 84)

$p \in V$, $w \in T_p(S_1)$ is represented by the vector $\alpha'(0)$ i.e. $w = \alpha'(0)$.

with $\alpha: (-\varepsilon, \varepsilon) \to V$, $\alpha(0) = p$.

Then $\beta := \varphi \circ \alpha$, $\beta(0) = \varphi(p)$. $\beta'(0) \in T_{\varphi(p)}(S_2)$

Now in our case, for $w \in T_p S$, take α as above, $\alpha'(0) = w$.

We define

$$df_p(w) = df_p(\alpha'(0)) := \beta'(0) \quad \text{where } \beta = f \circ \alpha.$$

More precisely:

$$df_p(w) = df_p(\alpha'(0)) = \left.\frac{d\beta}{dt}\right|_{t=0} = \left.\frac{d(f \circ \alpha(t))}{dt}\right|_{t=0}$$

$\boxed{\text{check this is well def. as Prop 2. (P84-85)}}$

$$= \left.\frac{d\,[(\alpha(t) - p_0)\cdot(\alpha(t) - p_0)]}{dt}\right|_{t=0}$$

$$= 2\alpha'(0)\,\alpha(0) - 2\alpha'(0)\,p_0$$

$$= 2\alpha'(0)\,[\alpha(0) - p_0]$$

$$= 2w \cdot (p - p_0)$$

// actually, we havent use any explicity form of $\alpha(t)$, we only use $\begin{cases} 1^\circ & \alpha(0) = p \\ 2^\circ & \alpha'(0) = w \end{cases}$

So the computation is independent of the choice of $\alpha(t)$.

$\square$

Contents P_{100}, #8, #9 ; P_{102} # 15 (if time possible)

P_{100} #8 Prove that whenever the coordinate curves constitute a Tchebyshef net it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first quadratic form are

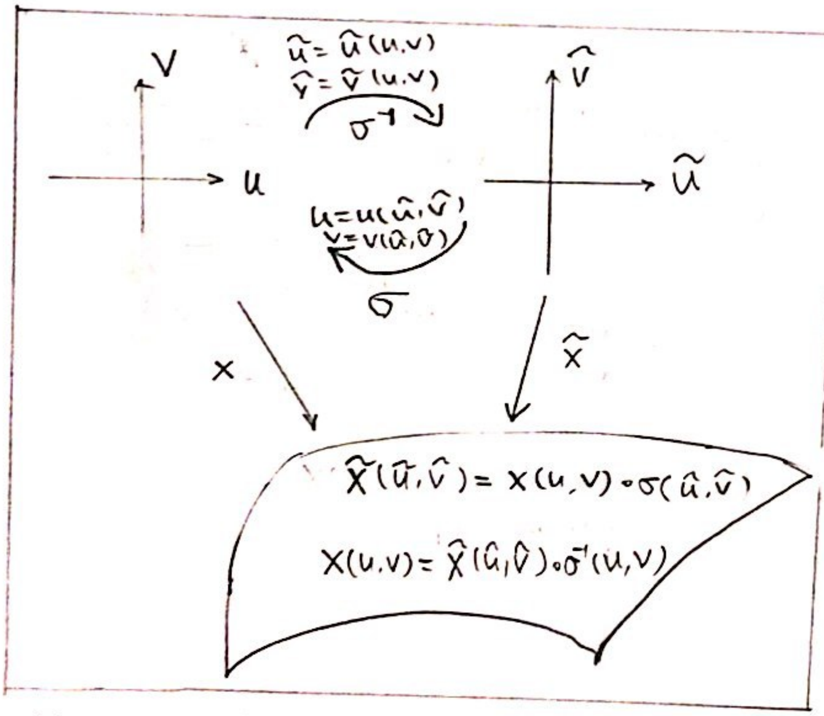
$$E = 1 \qquad F = \cos\theta, \qquad G = 1.$$

Recall Tchebyshef net (P_{100}, #7) $\Longleftrightarrow$ $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$

Since $\frac{\partial E}{\partial v} = 0$, $E = E(u) = X_u \cdot X_u$; Similarly $G = G(v)$

We want to reparametrize the coordinate, i.e. find

$\tilde{X} = \tilde{X}(\hat{u}, \hat{v})$; $\hat{u} = \hat{u}(u, v)$, $\hat{v} = \hat{v}(u, v)$ s.t. $\tilde{X}_{\hat{u}} \cdot \tilde{X}_{\hat{u}} = 1$, $\tilde{X}_{\hat{v}} \cdot \tilde{X}_{\hat{v}} = 1$, $\tilde{X}_{\hat{u}} \cdot \tilde{X}_{\hat{v}} = \cos\theta$



Disscussion :

$$\tilde{X}_{\hat{u}} = \frac{\partial \tilde{X}(\hat{u},\hat{v})}{\partial \hat{u}} = \frac{\partial (X(u,v) \circ \sigma(\hat{u},\hat{v}))}{\partial \hat{u}}$$

$$= \frac{\partial X(u(\hat{u},\hat{v}), v(\hat{u},\hat{v}))}{\partial \hat{u}}$$

$$= \frac{\partial X}{\partial u}\frac{\partial u}{\partial \hat{u}} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial \hat{u}}$$

$$= X_u \frac{\partial u}{\partial \hat{u}} + X_v \frac{\partial v}{\partial \hat{u}}$$

Similarly

$$X_{\hat{v}} = X_u \frac{\partial u}{\partial \hat{v}} + X_v \frac{\partial v}{\partial \hat{v}}$$

In sum

$$\boxed{\begin{aligned} \tilde{X}_{\hat{u}} &= X_u \frac{\partial u}{\partial \hat{u}} + X_v \frac{\partial v}{\partial \hat{u}} \\ \tilde{X}_{\hat{v}} &= X_u \frac{\partial u}{\partial \hat{v}} + X_v \frac{\partial v}{\partial \hat{v}} \end{aligned}} \quad \boxed{\begin{aligned} X_u &= \tilde{X}_{\hat{u}} \frac{\partial \hat{u}}{\partial u} + \tilde{X}_{\hat{v}} \frac{\partial \hat{v}}{\partial u} \\ X_v &= \tilde{X}_{\hat{u}} \frac{\partial \hat{u}}{\partial v} + \tilde{X}_{\hat{v}} \frac{\partial \hat{v}}{\partial u} \end{aligned}}$$

or

$$\begin{pmatrix} \tilde{X}_{\hat{u}} \\ \tilde{X}_{\hat{v}} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial \hat{u}} & \frac{\partial v}{\partial \hat{u}} \\ \frac{\partial u}{\partial \hat{v}} & \frac{\partial v}{\partial \hat{v}} \end{pmatrix} \begin{pmatrix} X_u \\ X_v \end{pmatrix} = \frac{\partial(u,v)}{\partial(\hat{u},\hat{v})} \begin{pmatrix} X_u \\ X_v \end{pmatrix} = J \begin{pmatrix} X_u \\ X_v \end{pmatrix}$$

$$\begin{pmatrix} X_u \\ X_v \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{u}}{\partial u} & \frac{\partial \hat{v}}{\partial u} \\ \frac{\partial \hat{u}}{\partial v} & \frac{\partial \hat{v}}{\partial v} \end{pmatrix} \begin{pmatrix} \tilde{X}_{\hat{u}} \\ \tilde{X}_{\hat{v}} \end{pmatrix} = \frac{\partial(\hat{u},\hat{v})}{\partial(u,v)} \begin{pmatrix} \tilde{X}_{\hat{u}} \\ \tilde{X}_{\hat{v}} \end{pmatrix} = J^{-1} \begin{pmatrix} X_{\hat{u}} \\ X_{\hat{v}} \end{pmatrix}$$

$$\left.\begin{aligned}
\tilde{E} &= \tilde{X}_{\hat u}\cdot\tilde{X}_{\hat u} = E\left(\frac{\partial u}{\partial \hat u}\right)^2 + 2F\left(\frac{\partial u}{\partial \hat u}\frac{\partial v}{\partial \hat u}\right) + G\left(\frac{\partial v}{\partial \hat u}\right)^2 \\
\hat{F} &= \tilde{X}_{\hat u}\cdot\tilde{X}_{\hat v} = E\left(\frac{\partial u}{\partial \hat u}\frac{\partial u}{\partial \hat v}\right) + F\left(\frac{\partial u}{\partial \hat u}\frac{\partial v}{\partial \hat v} + \frac{\partial v}{\partial \hat u}\frac{\partial u}{\partial \hat v}\right) + G\left(\frac{\partial v}{\partial \hat u}\frac{\partial v}{\partial \hat v}\right) \\
\hat{G} &= \tilde{X}_{\hat v}\cdot\tilde{X}_{\hat v} = E\left(\frac{\partial u}{\partial \hat v}\right)^2 + 2F\left(\frac{\partial u}{\partial \hat v}\frac{\partial v}{\partial \hat v}\right) + G\left(\frac{\partial v}{\partial \hat v}\right)^2
\end{aligned}\right] \quad (\star)$$

i.e. $\begin{pmatrix}\hat E & \hat F\\ \hat F & \tilde G\end{pmatrix} = J\begin{pmatrix}E & F\\ F & G\end{pmatrix}J^T$, $\quad J := \dfrac{\partial(u,v)}{\partial(\hat u,\hat v)} = \begin{pmatrix}\frac{\partial u}{\partial \hat u} & \frac{\partial v}{\partial \hat u}\\ \frac{\partial u}{\partial \hat v} & \frac{\partial v}{\partial \hat v}\end{pmatrix}$

Recall also
$$\left.\begin{aligned}
du &= \frac{\partial u}{\partial \hat u}d\hat u + \frac{\partial u}{\partial \hat v}d\hat v\\
dv &= \frac{\partial v}{\partial \hat u}d\hat u + \frac{\partial v}{\partial \hat v}d\hat v
\end{aligned}\right\} \Rightarrow \text{ i.e. } (du, dv) = (d\hat u, d\hat v)\,J$$

If you define the differential 2-form I by
$$I(u,v) = E\,du\,du + 2F\,du\,dv + G\,dv\,dv = (du, dv)\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}du\\ dv\end{pmatrix}$$

Then you will find

$$\begin{aligned}
I(\hat u,\hat v) &= (d\hat u, d\hat v)\begin{pmatrix}\hat E & \tilde F\\ \hat F & \tilde G\end{pmatrix}\begin{pmatrix}d\hat u\\ d\hat v\end{pmatrix}\\
&= (d\hat u, d\hat v)\,J\begin{pmatrix}E & F\\ F & G\end{pmatrix}J^T\begin{pmatrix}d\hat u\\ d\hat v\end{pmatrix}\\
&= (du, dv)\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}du\\ dv\end{pmatrix}\\
&= I(u,v),
\end{aligned}$$

i.e. this two-form is independent of the choice of parametrization.

end of discussion.

Now go back to #8, $E = E(u)$, indep. of v, $G = G(v)$

$$\hat E = E(u)\left(\frac{\partial u}{\partial \hat u}\right)^2 + 2F\frac{\partial u}{\partial \hat u}\frac{\partial v}{\partial \hat u} + G(v)\left(\frac{\partial v}{\partial \hat u}\right)^2 \overset{?}{=} 1$$

Try $\hat u := \int\sqrt{E(u)}\,du$, $\quad \hat v := \int\sqrt{G(v)}\,dv$, $\quad$ i.e. $\dfrac{\partial \hat u}{\partial u} = \sqrt{E(u)}$, $\quad \dfrac{\partial \hat v}{\partial u} = 0$

$\dfrac{\partial \hat u}{\partial v} = 0$ $\quad \dfrac{\partial \hat v}{\partial v} = \sqrt{G(v)}$

Then $\tilde E \overset{(\star)}{=} E(u)\cdot\dfrac{1}{E(u)} + 0 + 0 = 1$, Similarly, $\hat{F} \overset{\circledast}{=} F\left(\frac{1}{\sqrt{E\cdot G}}\right) = \frac{\langle X_u, X_v\rangle}{|X_u|\,|X_v|} = \cos\theta$ (P95)

$\hat{G} \overset{\circledast}{=} G\,\frac{1}{\sqrt{G}\cdot\sqrt{G}} = 1$

angle between X_u & X_v.

Therefore:

$$\underbrace{\frac{\langle X_{\hat u}, \hat{X}_{\hat v}\rangle}{|X_{\hat u}|\,|X_{\hat v}|}}_{\cos(\text{angle between } X_{\hat u} \,\&\, X_{\hat v})} = \frac{\hat{F}}{\sqrt{\hat E \hat G}} = \tilde{F} \xlongequal[\text{computation}]{\text{by above}} \frac{F}{\sqrt{E\cdot G}} = \underbrace{\frac{\langle X_u, X_v\rangle}{|X_u|\,|X_v|}}_{\cos(\text{angle between } X_u \,\&\, X_v)}$$

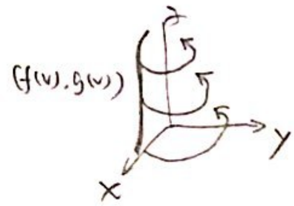
θ is also the angle between $X_{\hat u}$ & $X_{\hat v}$.

$\square$

P100

#9 Show that the surface of revolution can always be parametrized so that

$E = E(v)$, $F = 0$, $G = 1$.



Recall P76 Example 4.

$X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$

f, g with some assumptions.

$X_u = (-f(v)\sin u,\ f(v)\cos u,\ 0)$

$X_v = (f'(v)\cos u,\ f'(v)\sin u,\ g'(v))$

$E = f^2(v)$ $\quad F = 0$ $\quad G = f'(v)^2 + g'(v)^2$

If $\hat{v} := \int \sqrt{f'(v)^2 + g'(v)^2}\, dv$, i.e. the original curve

$\hat{u} := u$ $\qquad \alpha: v \to (f(v), g(v))$

is reparametrized by arclength $\hat{v}$.,

then by computation, we will find

$\frac{\partial \hat v}{\partial v} = \sqrt{G}$ $\quad \frac{\partial \hat u}{\partial u} = 1$, $\quad \frac{\partial \hat v}{\partial u} = 0$, $\quad \frac{\partial \hat u}{\partial v} = 0$

$\hat{E} \overset{\circledast}{=} f^2(v)$ ← only dep on $\hat v$, $\quad \hat{F} \overset{\circledast}{=} 0$

$\hat{G} = G\cdot\left(\frac{1}{\sqrt{G}}\right)^2 = 1$.

$\square$

P102 # 15. (Orthogonal Families of Curves)

(a) $X: U \subset R^2 \to S$ with E, F, G defined in the book

Let $\varphi(u,v) = \text{const.}$ $\psi(u,v) = \text{const.}$ be two families of regular curves on $X(U) \subset S$. (see P92 exercise 28 for regular curve)

Prove that these two families are orthogonal if & only if

$$E\varphi_v\psi_v - F(\varphi_u\psi_v + \varphi_v\psi_u) + G\varphi_u\psi_u = 0 \quad (*)$$

(b) Apply part (a) to show that on the coordinate neighborhood $X(U)$ of the helicoid of Example 3 (P94) the two families of regular curves

$$v\cos u = \text{const.} \qquad v \neq 0$$

$$(v^2+a^2)\sin^2 u = \text{const.} \qquad v \neq 0 \quad u \neq \pi$$

(Please check they are regular)

are orthogonal.

We only prove part (a). for part (b) you only need to parametrize helicoid, compute it E, F, G, and check the relation (*) holds for the given two families of regular curves.

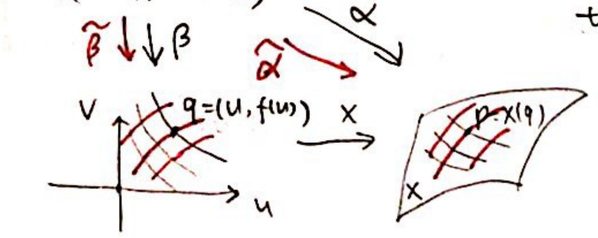
$$\varphi(u,v) = \text{const.} \overset{\text{implicity fun.}}{\underset{\text{thm}}{\Longleftrightarrow}} \quad v = f(u) \quad \& \quad \varphi(u, f(u)) = \text{const.}$$

(why you can use this?)

$$\Leftrightarrow \frac{d}{du}(\varphi(u, f(u))) = 0, \quad \text{i.e.} \quad \varphi_u + \varphi_v f'(u) = 0. \quad (1)$$

Similarly, for family ψ, we assume $v = g(u)$. then $\psi_u + \psi_v g'(u) = 0$ (2)

The diagram in P84 of book can now be simplified as below,



i.e. we use u as parameter of the first family curve α.

then
$$\alpha'(u) = \frac{d}{du}(X \circ \beta)(u)$$
$$= X_u(q)\,u'(u) + X_v(q)\,v'(u)$$
$$= X_u(q)\cdot 1 + X_v(q)\cdot f'(u)$$

1. Consider the surface S given by $y = zx$. Chose a parameterization of the surface S and compute the first fundamental form of the surface under the parameterization. Find points on the surface where the coordinate curves under your parameterization are orthogonal.

Sol.

Let's consider $\vec{X} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$$(x, z) \mapsto (x, zx, z)$$

Then by Proposition 1 of page 58, the graph of $y = y(x,z) = zx$ which is given by the above parametrization $\vec{X}$, is a regular surface.

Prop 1. If $f: U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x,y))$ for $(x,y) \in U$, is a regular surface.
Here we take $U = \mathbb{R}^2$, $f: U \to \mathbb{R}$ to be $y = y(x,z) = zx$, z, x as parameters

$\vec{X}_x = (1, z, 0)$, $\vec{X}_z = (0, x, 1)$

$E = \vec{X}_x \cdot \vec{X}_x = 1+z^2$, $F = \vec{X}_x \cdot \vec{X}_z = zx$, $G = \vec{X}_z \cdot \vec{X}_z = x^2+1$

So $\quad I = (1+z^2)\,dx^2 + 2zx\,dx\,dz + (1+x^2)\,dz^2$

Since the angle φ of the coordinate curves of a parametrization $\vec{X}(x,z)$ is
$$\cos\varphi = \frac{F}{\sqrt{EG}} \quad ;$$
it follows that the coordinate curves under above parametrization are orthogonal if & only if $F(x,z) = 0$ for all (x,z).

i.e. $zx = 0 \Leftrightarrow x = 0$ or $z = 0$. Those points are $\{(0,0,z) \mid z \in \mathbb{R}^1\} \cup \{(x,0,0) \mid x \in \mathbb{R}^1\}$ on the surface S.

2 Let $\alpha(s)$ be a regular space curve in $\mathbb{R}^3$ parametrized by the arclength parameter s. Suppose the curvature $\kappa(s)$ of the curve is non-zero everywhere and the binormal vector $\vec{b}(s) = \frac{\sqrt{2}}{2}(1, \cos s, \sin s)$. Compute $\kappa(s)$ & $|\tau(s)|$.

Remark. please see Tutorial 3 notes for the general computation. (P26 #15)

Sol. Since $\alpha(s)$ is parametrized by the arclength s, we have

① $\alpha' = t$ & $|t| = 1$ (arclength)

② $t' = \kappa n$

③ $n' = -\kappa t - \tau b$

④ $b' = \tau n$

$$|\tau| = |b'| = \left|\left(\frac{\sqrt{2}}{2}(1, \cos s, \sin s)\right)'\right| = \left|\frac{\sqrt{2}}{2}(0, -\sin s, \cos s)\right|$$

$$= \frac{\sqrt{2}}{2}\sqrt{(-\sin s)^2 + (\cos s)^2} = \frac{\sqrt{2}}{2}$$

$n = \frac{1}{\tau} b'$, so either $\tau = \frac{\sqrt{2}}{2}$, $n = (0, -\sin s, \cos s)$ ..... case A

or $\tau = \frac{-\sqrt{2}}{2}$, $n = (0, \sin s, -\cos s)$ ..... case B

In case A. $n' = (0, -\cos s, \sin s) = -\kappa t - \tau b$

so $-\kappa t = n' + \tau b = (0, -\cos s, -\sin s) + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}(1, \cos s, \sin s)$

$$= \left(\frac{1}{2}, -\frac{1}{2}\cos s, \frac{1}{2}\sin s\right)$$

$\Rightarrow$ $|-\kappa t| = \sqrt{\left(\frac{1}{2}\right)^2 + \frac{1}{4}(\cos s)^2 + \frac{1}{4}(\sin s)^2} = \frac{\sqrt{2}}{2}$

i.e. $\kappa = \frac{\sqrt{2}}{2}$ (recall $\kappa > 0$, $|t| = 1$)

Similarly, in case B, $n' = (0, \cos s, \sin s)$

$$-\kappa t = n' + \tau b = (0, \cos s, \sin s) + \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2}(1, \cos s, \sin s)$$

$$= \left(-\frac{1}{2}, \frac{1}{2}\cos s, \frac{1}{2}\sin s\right)$$

$\Rightarrow$ $|-\kappa t| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\cos s\right)^2 + \left(\frac{1}{2}\sin s\right)^2}$

$\Rightarrow$ $\kappa = \frac{\sqrt{2}}{2}$

In sum, $|\tau(s)| = \frac{\sqrt{2}}{2}$, $\kappa(s) = \frac{\sqrt{2}}{2}$ $\square$

MATH 321 Tutorial 8 P151 #3, #9(a)

Review

1. curvature of a curve $k \geq 0$

2. different kinds of curvature on surface

① eigenvalue of linear map dN_p. $-k_1$, $-k_2$

$k_1 \geq k_2$, principal curvatures (P144)

② normal curvature of $C \subset S$ at p (P141)

$k_n = k\cos\theta$, $\cos\theta = \langle n, N\rangle$. k: curvature of C.

③ Gauss curvature $K = k_1 k_2$

④ Mean curvature $H = \frac{1}{2}(k_1 + k_2)$

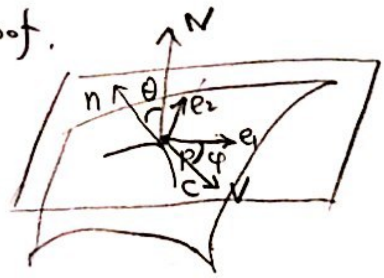
3. Geometric meaing of above curvatures.

4. Computation methods of above curvatures

#3 (P151) Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$k \geq \min(|k_1|, |k_2|).$$

Proof.



$k_n = k\cos\theta$ θ = angle btw n & N

φ = angle btw v & e_1

$k_n = k_1\cos^2\varphi + k_2\sin^2\varphi$ (Euler formula)

$k_1 \geq k_2$,

We know $K = k_1 k_2 > 0$, so either $k_1 \geq k_2 > 0$ or $0 > k_1 \geq k_2$

case ① $k_1 \geq k_2 > 0$, $\min(|k_1|, |k_2|) = k_2$, we need to show $k \geq k_2$

Now we know $k_n = k_1\cos^2\varphi + k_2\sin^2\varphi \geq k_2\cos^2\varphi + k_2\sin^2\varphi = k_2 > 0$

i.e. $k\cos\theta = k_2 > 0$

By the definition of k, $k \geq 0$, so $1 \geq \cos\theta > 0$, and we have

$$k = \frac{k_2}{\cos\theta} \geq \frac{k_2}{1} = k_2.$$

Case② $0 > k_1 \geq k_2$. Then $\min(|k_1|, |k_2|) = -k_1$

we need to show $k > -k_1$.

Now we have

$$k_n = k_1\cos^2\varphi + k_2\sin^2\varphi \leq k_1\cos^2\varphi + k_1\sin^2\varphi = k_1 < 0$$

i.e. $k_n = k\cos\theta < 0$ Therefore $-1 \leq \cos\theta < 0$.

$k\cos\theta \leq k_1$, i.e. $k(-\cos\theta) \geq -k_1 > 0$

$$\Rightarrow \quad k \geq \frac{(-k_1)}{(-\cos\theta)} \geq \frac{-k_1}{1} = -k_1$$

In sum, we have $k \geq \min(|k_1|, |k_2|)$ $\square$

P151

#9 (a). Prove that the image $N\circ\alpha$ by the Gauss map $N: S \to S^2$ of a parametrized regular curve $\alpha: I \to S$ which contains no planar or parabolic points is a parametrized regular curve on the sphere S^2 (called the spherical image of α).

(P136 Before Def. 1 : Throughout this chapter, S will denote a regular orientable surface in which an orientation has been chosen.)

$$\underset{\text{regular curve}}{I \xrightarrow{\alpha}} \underset{\text{contains NO planar or parabolic points}}{S} \xrightarrow{N} S^2 \qquad \text{i.e. } N\circ\alpha : I \to S^2$$

i.e. $\det(dN_p) \neq 0$.

Need to show $N\circ\alpha: I \to S^2$ is a $\underbrace{\text{parametrized regular curve}}$ (on the sphere S^2)

i.e. $(N\circ\alpha)'(t) \neq 0 \quad \forall t \in I$

Proof : $\forall t \in I$, we have a map

$$N \circ \alpha : I \to S^2$$
$$t \mapsto N(\alpha(t))$$

The smoothness of $(N \circ \alpha)$ is guaranteed by the composition law.

We need to show $(N \circ \alpha)'(t) \neq 0 \quad \forall t \in I$.

since $\alpha : I \to S$ is a regular curve.

we have $\alpha'(t) \neq 0$ and $\alpha'(t) \in T_p S$

$dN_p : T_p S \longrightarrow T_{N(p)} S^2$ is a linear map.

Since $\det(dN_p) \neq 0$, dN_p is an isomorphism of linear space $T_p S$ and linear space $T_{N(p)} S^2$. (You can also choose bases for vector spaces $T_p S$ and $T_{N(p)} S^2$, then the linear map dN_p is represented by a 2×2 matrix, and determinent of this matrix is nonzero.)

Therefore $(dN_p)(\alpha'(t)) \neq 0$. $(dN_p)(\alpha'(t)) \in T_{N(p)} S^2$.

In sum, for any $t \in I$, we denote $p = \alpha(t)$ and have

$$I \xrightarrow{\alpha} S \xrightarrow{N} S^2$$

$$I \xrightarrow{\alpha'} T_p S \xrightarrow{dN_p} T_{N(p)} S^2$$

$$(N \circ \alpha)'(t) = dN_p \circ \alpha'(t) = dN_p(\alpha'(t)) \neq 0 \quad \forall t$$

chain rule.

□

#14 If the surface S_1 intersects the surface S_2 along the regular curve C, then the curvature k of C at $p \in C$ is given by

$$k^2 \sin^2\theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\theta, \quad (*)$$

where λ_1 and λ_2 are the normal curvatures at P, along the tangent line to C, of S_1 and S_2, respectively, and θ is the angle made up by the normal vectors of S_1 and S_2 at P.

Analysis:

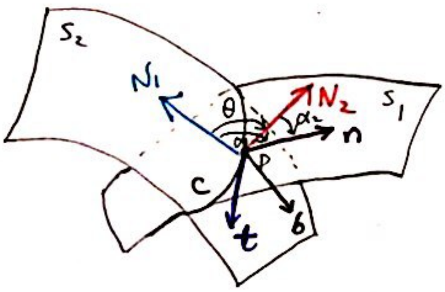
k — curvature of curve C

λ_i — normal curvature of S_i at P along the tangent line to C.

θ — angle btw N_1 & N_2, N_i normal vector of S_i at P.

Question: What's the relation between k & λ_i

Recall ① $\boxed{\lambda_i = k\cos\alpha_i}$, where $\alpha_i = \langle N_i, n\rangle$ angle between N_i & normal vector of curve C



Let t be tangent vector of C at p

t b n Frenet trihedron
(actually, we don't need b here).

So $t \perp b$, $t \perp n$

since t is on the tangent space of S_i at P,

$t \perp N_i$.

Therefore N_1, N_2, n, & b are in the same plane. In that plane, We have above angles, actually, we can refine those angles to be "directed angle". i.e. $\theta :=$ angle from N_1 to N_2 (not from N_2 to N_1)
(or oriented angle)
$\alpha_i =$ angle from N_i to N.

Then: $\boxed{\alpha_1 - \alpha_2 = \theta}$ ②

From box ① & ②, and using some knowledge of trigonometric functions, You can obtain the conclusion by yourself.

By putting ① into ⊛, what we need to prove is that

$$\sin^2\theta \overset{?}{=} \cos^2\alpha_1 + \cos^2\alpha_2 - 2\cos\alpha_1\cos\alpha_2\cos\theta$$

(Recall C is a regular curve, so $k>0$. We divide k^2 from ⊛)

where $\boxed{\theta = \alpha_1 - \alpha_2}$ ②

$$LHS = \sin^2(\alpha_1-\alpha_2) = (\sin\alpha_1\cos\alpha_2 - \cos\alpha_1\sin\alpha_2)^2$$

$$RHS = \cos^2\alpha_1 + \cos^2\alpha_2 - 2\cos\alpha_1\cos\alpha_2\underbrace{(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)}_{\text{Recall the formula for } \cos(\alpha_1-\alpha_2)}$$

$$= (\cos^2\alpha_1 - \cos^2\alpha_1\cos^2\alpha_2) + (\cos^2\alpha_2 - \cos^2\alpha_1\cos^2\alpha_2) - 2\cos\alpha_1\cos\alpha_2\sin\alpha_1\sin\alpha_2$$

$$= \cos^2\alpha_1\sin^2\alpha_2 + \cos^2\alpha_2\sin^2\alpha_1 - 2(\cos\alpha_1\sin\alpha_2)(\cos\alpha_2\sin\alpha_1)$$

$$= (\cos\alpha_1\sin\alpha_2 - \cos\alpha_2\sin\alpha_1)^2$$

$\Rightarrow$ LHS = RHS

□

Remark: You can use the hint in the textbook for another proof.

15. (Theorem of Joachimstahl.) Suppose that S_1 & S_2 intersect along a regular curve C and make an angle $\theta(p)$, $p\in C$. Assume that C is a line of curvature of S_1.
Prove that $\theta(p)$ is constant iff C is a line of curvature of S_2.

Recall def. (P145) If a regular connected curve C on S is such that for all $p\in C$ the tangent line of C is a principal direction at P, then C is said to be a line of curvature of S.

Also, we have a criterion :

Prop 3 (P145) (Olinde Rodrigues) A necessary and sufficient condition for a connected regular curve C on S to be a line of curvature of S is that

$$N'(t) = \lambda(t)\,\alpha'(t) \quad , \quad \text{where} \quad N(t) = N \circ \alpha(t)$$

$\alpha(t)$ any parametrization of C

$\lambda(t)$ differentiable function of t.

In this case, $-\lambda(t)$ is the principal curvature along $\alpha'(t)$.

Analysis :

Let $N_{i,p}$ be normal vector of S_i at $p \in C$, and assume that is represented by $\alpha : I \to R^3$ & $\alpha(t) = p$.

We know C is a line of curvature of S_1, i.e.

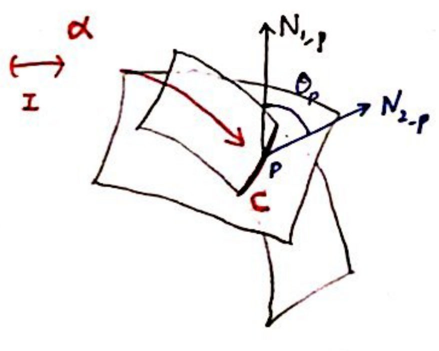
$$N_{1,p}'(t) = \lambda(t)\,\alpha'(t) \quad , \quad N_{1,p}(t) = N_{1,p} \circ \alpha(t)$$

$\theta(p) = \text{angle}\langle N_{1,p}, N_{2,p} \rangle$, angle between $N_{1,p}$ & $N_{2,p}$

We want to show

$$\theta(p) = \text{const.} \iff N_{2,p}'(t) = \tilde{\lambda}(t)\,\alpha'(t)$$

for some differentiable function $\tilde{\lambda}(t)$



$\theta(p) = \text{const.}$ angle between them

$\text{angle}\langle N_{1,p}, N_{2,p} \rangle = \theta(p)$ independent of t

$$\iff (N_{1,p} \circ \alpha(t), N_{2,p} \circ \alpha(t)) = \text{const.}$$

inner product in R^3

$$\iff \frac{d}{dt} (N_{1,p}(t), N_{2,p}(t)) = 0$$

$$\iff N_{1,p}'(t) \cdot N_{2,p}(t) + N_{1,p}(t) \cdot N_{2,p}'(t) = 0 \qquad (*)$$

inner product.

Proof.

"⇒" If C is a line of curvature of S_1, i.e

$$N_{1,p}'(t) = \lambda(t)\,\alpha'(t). \quad ①$$

then by ⊛, (i.e. $\theta(p) = $ const.)

$$\lambda(t)\,\alpha'(t)\cdot N_{2,p}(t) + N_{1,p}(t)\cdot N_{2,p}'(t) = 0$$

Since $\alpha'(t)$ is the tangent direction of C, it is in the tangent space of S_2 of p. Then

$\alpha'(t) \perp N_{2,p}(t)$. So $\alpha'(t)\cdot N_{2,p}(t) = 0$.

Then we have

$$N_{1,p}(t)\cdot N_{2,p}'(t) = 0.$$

i.e. $N_{2,p}'(t) \perp N_{1,p}(t)$ ②

$$|N_{2,p}(t)| = 1 \Rightarrow N_{2,p}(t)\cdot N_{2,p}(t) = 1$$

$$\Rightarrow N_{2,p}(t)\cdot N_{2,p}'(t) = 0$$

$$\Rightarrow N_{2,p}'(t) \perp N_{2,p}(t) \quad ③$$

- If $\theta(p) = $ const $\neq 0$, i.e. $N_{1,p}(t) \neq N_{2,p}(t)$,

  then $N_{1,p}(t)$ & $N_{2,p}(t)$ span a space, and by ② & ③

  $N_{2,p}'(t)$ is the normal vector of the space

  i.e. $N_{2,p}'(t) \parallel \alpha'(t)$

  So we can write $N_{2,p}'(t) = \hat{\lambda}(t)\,\alpha'(t)$ ④ for some differentiable function $\hat{\lambda}(t)$ of t.

  i.e. C is also a line of curvature of S_2.

- If $\theta(p) = $ const. $= 0$, i.e. $N_{1,p}(t) = N_{2,p}(t)$,

  then ① also tells us C is a line of curvature of S_2

"⟸" Conversely, if C is also a line of curvature of S_2, then we have formula ④ in page ④.

Formulae ① & ④ in page ④ ⟹ ✱ in page ③

$\Rightarrow \theta(p) = \text{const}.$ $\square$

#17 (P152) ① Show that if $H \equiv 0$ on S and S has no planar points, then the Gauss map $N: S \to S^2$ has the following property:

$$\langle dN_p(w_1), dN_p(w_2) \rangle = -K(p) \langle w_1, w_2 \rangle \quad ✱$$

for all $p \in S$ and all $w_1, w_2 \in T_p(S)$.

② show that the angle of two intersecting curves on S and the angle of their spherical image (cf. Exer. 9) are equal up to a sign.

Analysis: $H = \frac{1}{2}(k_1 + k_2)$.

No planar points $\Rightarrow dN_p \neq 0 \quad \forall p$

so we exclude the case $k_1 = k_2 = 0$.

Since $H \equiv 0$, we must have $k_1 > 0$, $k_2 = -k_1 < 0$

Let e_i be the normal eigenvector corresponding to k_i, i.e. $dN_p(e_i) = k_i e_i$, $|e_i| = 1$

then $e_1 \perp e_2$. (why ?) and they span the tangent space of S at p, so for any $w_1, w_2 \in T_p S$, we write

$$\begin{aligned} w_1 &= a e_1 + b e_2 \\ w_2 &= c e_1 + d e_2 \end{aligned} \Rightarrow \begin{aligned} dN_p(w_1) &= a k_1 e_1 + b k_2 e_2 \\ dN_p(w_2) &= c k_1 e_1 + d k_2 e_2 \end{aligned}$$

$$\langle dN_p(w_1), dN_p(w_2) \rangle = ac\, k_1^2 + bd\, k_2^2 = -k_1 k_2 (ac + bd) = -K(p) \langle w_1, w_2 \rangle$$

For question ②, we assume two intersecting curves have tangent direction w_1 and w_2, since we only concern the angle between w_1 and w_2, we can assume $|w_1| = |w_2| = 1$

Then $\cos\theta = \langle w_1, w_2 \rangle$. For the spherical image, $\cos \overset{\text{angle btw spherical image}}{\langle \text{angle} \rangle} = \dfrac{\langle dN_p(w_1), dN_p(w_2) \rangle}{|dN_p(w_1)|\,|dN_p(w_2)|}$

then by ✱ $\cos\langle \text{angle} \rangle = \dfrac{-K(p) \langle w_1, w_2 \rangle}{|K(p)|\,|w_1|\,|w_2|} = \cos\theta \Rightarrow \text{angle} = \pm\theta$ $\square$

$(K(p) = k_1 k_2 < 0)$

Home work

P151 #2 $C \subset S$

φ

$\alpha(t)$.

N is not a function of t originally

Normal vector $\vec{N}(x,y,z)$ of S is constant along $\alpha(t)$

not $N(t) \cdot \alpha(t)$

So $\vec{N} \circ (\alpha(t)) = \vec{N}(\alpha(t)) = \text{const}.$

chain rule

$\Rightarrow 0 = \frac{d}{dt} \vec{N}(\alpha(t)) = dN_{\alpha(t)}(\alpha'(t))$

i.e. $dN_{\alpha(t)}(\alpha'(t)) = 0 \ \alpha'(t)$

$\Rightarrow$ 0 is eigenvalue

$\Rightarrow$ planar or parabolic

#2. Determine the asymptotic curves and the lines of curvature of helicoid $x = v\cos u,\ y = v\sin u,\ z = cu$, and show that its mean curvature is zero.
constant, $c \neq 0$.

Recall

① P148 Def 9. $p \in S$. An asymptotic direction of S at p is a direction of T_pS for which the normal curvature is zero. An asymptotic curve of S is a regular connected curve $C \subset S$ s.t. $\forall p \in C$ the tangent line of C at p is an asymptotic direction.

② P160 A connected regular curve C in the coordinate neighborhood of $X(u,v)$ is an asymptotic curve iff for any parametrization $\alpha(t) = X(u(t), v(t)),\ t \in I$, of C we have $\mathrm{II}(\alpha'(t)) = 0,\ \forall t \in I$,

i.e. iff $e(u')^2 + 2fu'v' + g(v')^2 = 0 \quad t \in I$. differential equation of the asymptotic curve

Now $X(u,v) = (v\cos u,\ v\sin u,\ cu)$

$X_u = (-v\sin u,\ v\cos u,\ c)$ $\qquad X_{uu} = (-v\cos u,\ -v\sin u,\ 0)$

$X_v = (\cos u,\ \sin u,\ 0)$ $\qquad X_{vv} = (0,\ 0,\ 0)$

$X_u \times X_v = (-c\sin u,\ c\cos u,\ -v)$ $\qquad X_{uv} = (-\sin u,\ \cos u,\ 0)$

$E = \langle X_u, X_u \rangle = v^2 + c^2 \qquad F = \langle X_u, X_v \rangle = 0 \qquad G = 1$

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{X_u \times X_v}{\sqrt{EG - F^2}} = \frac{1}{\sqrt{v^2+c^2}}(-c\sin u,\ c\cos u,\ -v)$$

$e = \langle N, X_{uu} \rangle = 0 \qquad f = \langle N, X_{uv} \rangle = \dfrac{c}{\sqrt{v^2+c^2}} \qquad g = \langle N, X_{vv} \rangle = 0$

Then the differential equation of the asymptotic curve is

$$\frac{2c}{\sqrt{v^2+c^2}} u'v' = 0 \overset{c\neq 0}{\Rightarrow} u' = 0 \text{ or } v' = 0 \Rightarrow \text{Asymptotic curves } u = \text{const. or } v = \text{const}$$

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} = 0.$$

(see P161) The differential equation of the lines of curvature

$$(fE - eF)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 = 0$$

In our case, the above equation reads:

$$\frac{c}{\sqrt{v^2+c^2}}(v^2+c^2)(u')^2 - \frac{c}{\sqrt{v^2+c^2}}(v')^2 = 0$$

i.e. $(v^2+c^2)(u')^2 = (v')^2$

$$u' = \frac{v'}{\sqrt{v^2+c^2}}$$

$$u(t) = \int \frac{v'(t)}{\sqrt{v^2+c^2}}\,dt + \text{Const.}$$

By standard calculation, (*) $\displaystyle\int \frac{v'(t)}{\sqrt{v^2+c^2}}\,dt = \pm \log_e(v+\sqrt{v^2+c^2}) + \text{Const.}$

so <u>line of curvature</u>.

$$u(t) \mp \log_e(v(t)+\sqrt{v(t)^2+c^2}) = \text{const.}$$

Remarks on calculation (*)

$$\int \frac{v'(t)}{\sqrt{v(t)^2+c^2}}\,dt = \int \frac{dv}{\sqrt{v^2+c^2}} \overset{v=\pm c\sinh(z)}{=\!=\!=} \pm\int \frac{c\cosh(z)\,dz}{\sqrt{c^2(\cosh(z))^2}} = \pm\frac{c}{|c|}z + \text{const.}$$

↑ check this box for this step

Recall $\sinh(z) = \frac{e^z - e^{-z}}{2} > 0$

$\cosh(z) = \frac{e^z + e^{-z}}{2} > 0$

$1 + (\sinh(z))^2 = (\cosh(z))^2$

$(\sinh(z))' = \cosh(z)$

$(\cosh(z))' = \sinh(z)$

$\frac{v}{c} = \sinh(z) = \frac{e^z - e^{-z}}{2} > 0$

$e^{2z} - 2\left(\frac{v}{c}\right)e^z - 1 = 0$

$e^z = \frac{2\left(\frac{v}{c}\right) + \sqrt{\left(2\left(\frac{v}{c}\right)\right)^2+4}}{2} = \frac{v}{c} + \sqrt{\left(\frac{v}{c}\right)^2+1}$

$z = \log_e\left(\frac{v}{c} + \sqrt{\left(\frac{v}{c}\right)^2+1}\right)$

If $c>0$, $z = \log_e\left(\frac{v}{c} + \frac{\sqrt{v^2+c^2}}{c}\right) = \log_e(v+\sqrt{v^2+c^2}) - \log_e c$

If $c<0$, $z = \log_e\left(\frac{v}{c} - \frac{\sqrt{v^2+c^2}}{c}\right) = \log_e(\sqrt{v^2+c^2} - v) - \log_e(-c)$

then $-z = -\log_e(\sqrt{v^2+c^2} - v) + \log_e(-c)$

$= \log_e(\sqrt{v^2+c^2} - v)^{-1} + \log_e(-c)$

$= \log_e\left[\frac{(v+\sqrt{v^2+c^2})}{c^2}\right] + \log_e(-c)$

$= \log_e(v+\sqrt{v^2+c^2}) - \log_e(-c)$

In sum $\frac{c}{|c|}z = \log_e(v+\sqrt{v^2+c^2}) - \log_e|c|$

↖ put this into const. term

$$\Rightarrow \int \frac{v'(t)}{\sqrt{v(t)^2+c^2}}\,dt = \pm\log_e(v+\sqrt{v^2+c^2}) + \text{Const} \quad (*)$$

12. Consider the parametrized surface

$$X(u,v) = (\sin u \cos v,\ \sin u \sin v,\ \cos u + \log \tan \tfrac{u}{2} + \varphi(v))$$

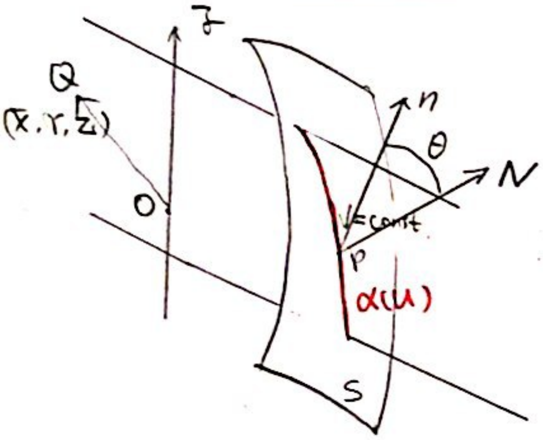
where φ is a differentiable function. Prove that

a. The curves $v = $ const. are contained in planes which pass through the z-axis and intersect the surface under a constant angle θ given by

$$\cos\theta = \frac{\varphi'}{\sqrt{1+(\varphi')^2}}.$$

Conclude that the curves $v=$ const are lines of curvature of the surface

b. The length of the segment of a tangent line to a curve $v=$const., determined by its point of tangency and the z-axis, is constantly equal to 1. Conclude that the curves $v=$ const. are <u>tractrices</u>. (see P_{168} Ex 6. a, P_8 Fig 1.9 and tutorial notes 2, Page 3)



- $v=$ const. this curve of course in S

need to show it also in a plane pass through z-axis

actually, this curve = intersection of the plane & S

- how to describe the intersection angle θ?

take $n=$ normal vector of the plane

$\cos\theta = \langle n, N_p\rangle$, where N_p is normal vector at pt p

- how to describe the plane passing through z-axis?

$$n = (\mu, \nu, 0) \qquad \mu^2+\nu^2=1$$

take any point Q on the plane, we call it (X, Y, Z)

then $\overrightarrow{OQ} = (X, Y, Z)$, where O is the origin

so $\overrightarrow{OQ}\cdot n = 0$ i.e. $\boxed{\mu X + \nu Y = 0}$

For curve $v=$ const. on the surface, i.e. points

$\alpha(u) = (\sin u\cos v,\ \sin u \sin v,\ \cos u + \log\tan\frac{u}{2} + \varphi(v))$, we call this u-curve (i.e. $v=$ const.)

we try to find $(\mu, \nu, 0)$ s.t their inner product is zero.

i.e. for $V = \text{const}$.

we need to find μ, ν s.t. ① $\mu^2 + \nu^2 = 1$

② $\sin u \cos v \cdot \mu + \sin u \sin v \, \nu = 0$. for any u

Actually, we can take $\underline{\mu = \sin v \quad \nu = -\cos v}$!

$\underline{n = (\sin v, -\cos v, 0)}$

Next, let's compute the normal vector for a point p on S

$$X_u = \left(\cos u \cos v, \ \cos u \sin v, \ \underbrace{-\sin u + \frac{1}{\tan\frac{u}{2}} \cdot \frac{1}{\cos^2\frac{u}{2}} \cdot \frac{1}{2}}\right)$$

$$X_v = (-\sin u \sin v, \ \sin u \cos v, \ \varphi'(v))$$

$$\frac{1}{2\sin\frac{u}{2}\cos\frac{u}{2}} = \frac{1}{\sin u}$$

$$X_u \times X_v = \left(\varphi'(v)\cos u \sin v - (\cos u)^2 \cos v, \ -\varphi'(v)\cos u \cos v - \sin v (\cos u)^2, \ \cos u \sin u\right)$$

$$= \cos u \left(\varphi'(v)\sin v - \cos u \cos v, -\varphi'(v)\cos v - \cos u \sin v, \ \sin u\right)$$

$$\cos\theta = n \cdot \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{\varphi'(v)\sin^2 v - \cos u \sin v \cos v + \varphi'(v)\cos^2 v + \cos u \sin v \cos v}{\sqrt{\begin{array}{l}(\varphi'(v)\sin v)^2 - 2\varphi'(v)\sin v \cos u \cos v + \cos^2 u \cos^2 v \\ + (\varphi'(v)\cos v)^2 + 2\varphi'(v)\cos v \cos u \sin v + \cos^2 u \sin^2 v + \sin^2 u\end{array}}}$$

$$= \frac{\varphi'(v)}{\sqrt{(\varphi'(v))^2 + 1}}$$

since $V = \text{const} \Rightarrow \cos\theta = \text{Const}$,

$\Rightarrow \theta$ const. angle.

where $N(u,v) = \dfrac{X_u \times X_v}{|X_u \times X_v|} = \dfrac{(\varphi'(v)\sin v - \cos u \cos v, \ -\varphi'(v)\cos v - \cos u \sin v, \ \sin u)}{\sqrt{1 + \varphi'(v)^2}}$

Next, let's try to conclude the u-curves (i.e. $V = \text{const}$.) are lines of curvature of the surface. We denote them by $\alpha(u)$

We want use "Olinde Rodrigues Thm" (P145), i.e.,
we want to find $\lambda(u)$ s.t.

$$\circledast \quad (N\circ\alpha(u))' = \lambda(u)\,\alpha'(u)$$

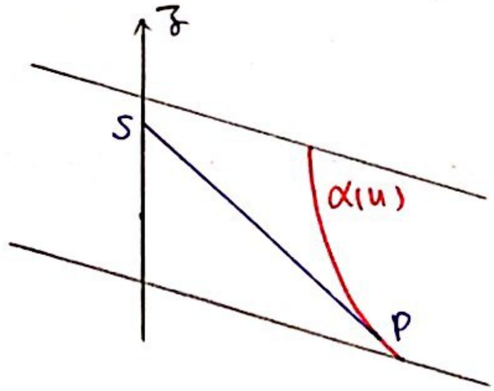
, where prime ' means taking derivative w.r.t. variable u

Actually $\underline{(N\circ\alpha(u))'} = N_u = \dfrac{(\sin u\cos v,\ \sin u\sin v,\ \cos u)}{\sqrt{\varphi'(v)^2+1}}$

$\underline{\alpha'(u)} = X_u = (\cos u\cos v,\ \cos u\sin v,\ -\sin u+\frac{1}{\sin u})$

Therefore we take $\underline{\lambda(u) := \dfrac{1}{\sqrt{\varphi'(v)^2+1}}\ \dfrac{\sin u}{\cos u}}$ and then $\circledast$ holds.

Let's consider part b.



$|PS| = 1$

$P:\ (\sin u\cos v,\ \sin u\sin v,\ \cos u+\log\tan\frac{u}{2}+\varphi(v))$

$S:\ (0,\ 0,\ w)$

- Write down the tangent line equation at point P, and find the coordinate w, Then compute $|PS|$.
- Or you can use $\overrightarrow{SP} \parallel X_u$

$$\overrightarrow{SP} = (\sin u\cos v,\ \sin u\sin v,\ \cos u+\log\tan\tfrac{u}{2}+\varphi(v)-w) = \underset{\text{Some constant.}}{k}\cdot\left(\cos u\cos v,\ \cos u\sin v,\ -\sin u+\tfrac{1}{\sin u}\right)$$

so $k = \dfrac{\sin u}{\cos u}$, and

$$\cos u+\log\tan\tfrac{u}{2}+\varphi(v)-w = \frac{\sin u}{\cos u}\left(-\sin u+\frac{1}{\sin u}\right) = \cos u$$

i.e. $\overrightarrow{SP} = (\sin u\cos v,\ \sin u\sin v,\ \cos u)$

$$\Rightarrow |\overrightarrow{SP}| = \sqrt{(\sin u\cos v)^2+(\sin u\sin v)^2+(\cos u)^2} = 1.$$

Try to conclude by yourself that $v=$ const. are tractrices. $\square$

P152 #10

$C \subset S \quad : \alpha(s) \qquad s$ arclength

$\alpha'(s) =$ principal direction $\Rightarrow \quad d\vec{N}_p(\alpha'(s)) = -k_i \alpha'(s)$ ①

$\alpha'(s)$ not asymptotic direction $\Rightarrow \quad \mathrm{II}(\alpha'(s)) \neq 0$

$(\mathrm{II}(v) = -\langle dN(v), v \rangle \quad v \in T_pS)$

$\therefore \quad -\langle dN_p(\alpha'(s)), \alpha'(s) \rangle \neq 0$ ②

$= -N' \cdot \alpha'(s) = -N' \cdot t$

$N \cdot t = 0 \Rightarrow \quad \overset{\Rightarrow}{=} N \cdot t' = kN \cdot n \neq 0 \quad k\cos\theta \neq 0 \Rightarrow k \neq 0, \ \cos\theta \neq 0$

① & ② $\Rightarrow \quad k_i |\alpha'(s)| \neq 0 \quad \Rightarrow \quad k_i \neq 0$

$k =$ curvature of C, Euler formula $k_i = k\cos\theta$ $\Big\} \Rightarrow k \neq 0$ i.e. $k > 0$

$\cos\theta \neq 0$

Then we try to show $\tau(s) = 0$ $\Big\} \Rightarrow$ C is plane curve
$(k > 0)$

osculating plane $\xrightarrow{\text{normal}} \vec{b}(s)$

Tangent plane of S along C $\Big] \xrightarrow{\text{normal}} N$

$b \cdot N = \text{const.}$

$b' \cdot N + b \cdot N' = 0 \qquad \boxed{N' = k_i \alpha'(s)}$

$b' \cdot N + f(s)\, b \cdot \underbrace{\alpha'(s)}_{t} = 0$

$\therefore \quad b' \cdot N = 0$

$b' = \tau(s)\, \vec{n}(s)$

$\tau(s) \cdot \underbrace{\cos\theta}_{\neq 0} = 0$

$\tau(s) = 0. \qquad \square$

P168 #3. Determine the asymptotic curves of the catenoid

$$X(u,v) = (\cosh v \cos u,\ \cosh v \sin u,\ v).$$

This question is similar to #2. Let's recall the differential equation of the asymptotic curves

$$\boxed{e(u')^2 + 2fu'v' + g(v')^2 = 0 \qquad t \in I}$$

$X_u = (-\cosh v \sin u,\ \cosh v \cos u,\ 0)$

$X_v = (\sinh v \cos u,\ \sinh v \sin u,\ 1)$

$X_{uu} = (-\cosh v \cos u, -\cosh v \sin u,\ 0)$

$X_{uv} = (-\sinh v \sin u, \sinh v \cos u,\ 0)$

$X_{vv} = (\cosh v \cos u,\ \cosh v \sin u,\ 0)$

$X_u \times X_v = (\cosh v \cos u,\ \cosh v \sin u,\ -\cosh v \sinh v)$

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(\cos u,\ \sin u,\ -\sinh v)}{\sqrt{1+\sinh^2 v}} = \frac{(\cos u,\ \sin u,\ -\sinh v)}{\cosh v}$$

$e = \langle N, X_{uu}\rangle = -1$, $f = \langle N, X_{uv}\rangle = 0$, $g = \langle N, X_{vv}\rangle = 1$

$$\boxed{-(u')^2 + (v')^2 = 0 \quad \forall\, t \in I}$$ i.e. $u'(t) = v'(t)$ or $u'(t) = -v'(t)$

$u = v + \text{const}$ or $u = -v + \text{const}.$

Please (use SAGE to) draw the geometric picture of this surface

P172 #13 Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be the map (a similarity) defined by $F(p) = cp$, $p \in \mathbb{R}^3$. (positive constant)

Let $S \subset \mathbb{R}^3$ be a regular surface and set $F(S) = \bar{S}$.

(1) show that $\bar{S}$ is a regular surface

(2) Find formulas relating the Gaussian & mean curvatures, K and H, of S with the Gaussian & mean curvatures, $\bar{K}$ and $\bar{H}$, of $\bar{S}$

(1) We can use the definition of regular surface (P52) and inverse function theorem (P131) to check that $\bar{S}$ is a regular surface.

Since $dF_p = c: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto cx$ is an isomorphism $\Rightarrow$ $F: \mathbb{R}^n \to \mathbb{R}^n$ is diffeomorphism

We know S regular, by definition (P52). $\forall p \in S$ $\exists$ neighborhood $V \subset \mathbb{R}^3$

$X: U \to S \cap V$ s.t. ① X is differentiable ② X is homeomorphism and $\forall q \in U$, $dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one

Then ★ $\bar{x}(u,v) := c\,x(u,v)$ will be a parametrization of $\bar{S}$.

Now let's check the definition of regular surface for $\bar{S}$:

$\forall \bar{p} \in \bar{S}$, $\bar{p} = cp$ for some $p \in S$. Since S is regular, $\exists$ nbhd V of p, then by diffeomorphism F, $F(V)$ is a nbhd of $\bar{p}$ in $\mathbb{R}^3$

$\bar{x} : U \to F(V) \cap \bar{S}$ is differentiable, a homeomorphism and

$d\bar{x}_q = c\,dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

(2) We can use ★ to compute $\bar{K}, \bar{H}$.

$\bar{x}_u = c\,x_u$, $\bar{x}_v = c\,x_v$, $\bar{x}_{uv} = c\,x_{uv}$, $\bar{N} = N$, $\bar{x}_{uu} = c\,x_{uu}$, $\bar{x}_{vv} = c\,x_{vv}$

$\Rightarrow \bar{E} = c^2 E$, $\bar{F} = c^2 F$, $\bar{G} = c^2 G$, $\bar{e} = ce$, $\bar{f} = cf$, $\bar{g} = cg$

$$\bar{K} = \frac{\bar{e}\bar{g} - \bar{f}^2}{\bar{E}\bar{G} - \bar{F}^2} = \frac{c^2(eg - f^2)}{c^4(EG - F^2)} = \frac{1}{c^2} K$$

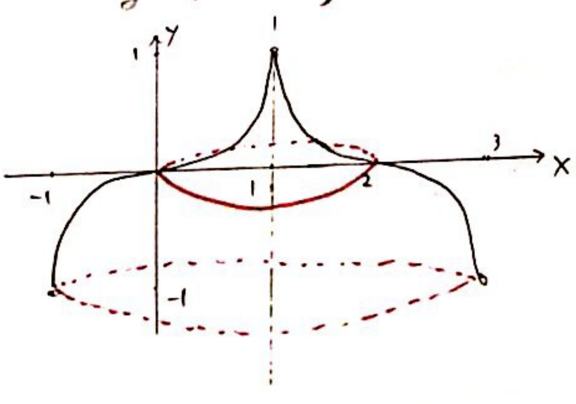
$$\bar{H} = \frac{1}{2}\frac{\bar{e}\bar{G} - 2\bar{f}\bar{F} + \bar{g}\bar{E}}{\bar{E}\bar{G} - \bar{F}^2} = \frac{c^3}{c^4}\frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} = \frac{1}{c} H$$

Remark: ① For sphere $x^2+y^2+z^2=1$ $K = 1$

$x^2+y^2+z^2 = r^2$ $K = \frac{1}{r^2}$

② K is 2-dimensional curvature; H is 1-dimensional curvature.

P172 #14. Consider the surface obtained by rotating the curve $y = x^3$, $-1<x<1$ about the line $x=1$. Show that the points obtained by rotation of the origin $(0,0)$ of the curve are planar points of the surface

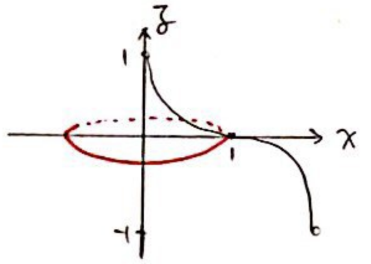


Intuitively $y = x^3$ is tangent to x-axis at $(0,0)$.

In order to use the computation results of Surface of Revolution (P76, P161)

(up to a rigid motion) take xz-plane as the plane of curve and the

z-axis as the rotation axis



$$x = v \qquad 0 < v < 2$$
$$z = -(v-1)^3$$

Recall $\left[\begin{array}{l} x = \varphi(v) \qquad a < v < b \\ z = \psi(v) \end{array}\right.$.

$$X(u,v) = (\, v\cos u, \ \ v\sin u, \ \ -(v-1)^3 \,)$$

$$U = \{(u,v) \in \mathbb{R}^2, \ 0 < u < 2\pi, \ 0 < v < 2\}$$

The curve is given by $X(u,1)$

We want to show $dN_{(u,1)} = 0 \quad \forall \ u \in (0, 2\pi)$

$$X_u = (-\varphi(v)\sin u, \ \varphi(v)\cos u, \ 0) \qquad \varphi(v) = v \qquad \varphi'(v) = 1$$

$$X_v = (\varphi'(v)\cos u, \ \varphi'(v)\sin u, \ \psi'(v)) \qquad \psi(v) = -(v-1)^3 \qquad \psi'(v) = -3(v-1)^2$$

$$\psi''(v) = -6(v-1)$$

$$X_u \times X_v = (\varphi(v)\psi'(v)\cos u, \ \varphi(v)\psi'(v)\sin u, \ -\varphi(v)\varphi'(v))$$

$$= (-3v(v-1)^2\cos u, \ -3v(v-1)^2\sin u, \ -v)$$

$$N(u,v) = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(3(v-1)^2\cos u, \ 3(v-1)^2\sin u, \ 1)}{\sqrt{9(v-1)^2+1}}$$

$$dN_{(u,1)} = N'(u,v)\Big|_{(u,v)=(u,1)} = \left.\frac{(-3(v-1)^2\sin u, \ 3(v-1)^2\cos u, \ 0)}{\sqrt{9(v-1)^2+1}}\right|_{(u,1)}$$

$$= \frac{(0, \ 0, \ 0)}{1} = \vec{0}$$

ie $\underline{dN_p = \vec{0} \quad \forall \ p \in \text{curve } X(u,1)}$

$\square$

Rmk ① We can parametrize the surface directly.

② Use results on P161-162, $K = -\dfrac{\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi} = \dfrac{-18(v-1)^2}{v} \qquad v \in (0,2)$

$$H = \frac{1}{2}\,\frac{-\psi' + \varphi(\psi'\varphi'' - \psi''\varphi')}{\varphi} = \frac{3(v-1)(v+1)}{2v}$$

Tutorial Notes 12 MATH321

Ch4 The intrinsic Geometry of surfaces

$\varphi: S \to \bar{S}$ diffeomorphism

Def. ① $\varphi: S \to \bar{S}$ isometry : $\forall p \in S$, $\forall w_1, w_2 \in T_pS$ we have

$$\langle w_1, w_2 \rangle_p = \langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle_{\varphi(p)}$$

(Recall $d\varphi_p : T_pS \to T_{\varphi(p)}\bar{S}$)

Def ② $\varphi: V \to \bar{S}$ of a nbhd V of $p \in S$ is a local isometry at p if

$\exists$ nbhd $\bar{V}$ of $\varphi(p) \in \bar{S}$ s.t. $\varphi: V \to \bar{V}$ is an isometry.

Def ③ S is locally isometric to $\bar{S}$ if there exists a local isometry into $\bar{S}$ at every $p \in S$

Def ④ S & $\bar{S}$ are locally isometric if $\begin{cases} S \text{ is locally isometric to } \bar{S} \\ \& \ \bar{S} \text{ - - - - - - - - } S \end{cases}$

Rmk: locally isometry $\not\Rightarrow$ global isometry

Criterion for local isometry (Prop 1, P220) (first fundamental forms are the same!)

Assume the existence of parametrizations $x: U \to S$ & $\bar{x}: U \to \bar{S}$ s.t. $E = \bar{E}$, $F = \bar{F}$ & $G = \bar{G}$ in U.

Then $\varphi := \bar{x} \circ x^{-1} : x(U) \to \bar{S}$ is a local isometry

$$\begin{tikzcd} U \arrow[r, "\bar{x}"] \arrow[d, shift right, "x"'] & \bar{S} \\ x(U) \subset S \arrow[u, shift right, "x^{-1}"'] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

"local" isometry

Exercises P228 #6, P229 #9, #10.

P228

#6. Let $\alpha: I \to \mathbb{R}^3$ be a regular parametrized curve with $k(t) \neq 0$, $t \in I$

Let $X(t, v)$ be its tangent surface. Prove that tangent surfaces are locally isometric to planes, i.e

- for each $(t_0, v_0) \in I \times (\mathbb{R} - \{0\})$, $\exists$ nbhd V of (t_0, v_0) s.t.

$X(V)$ is isometric to an open set of the plane.

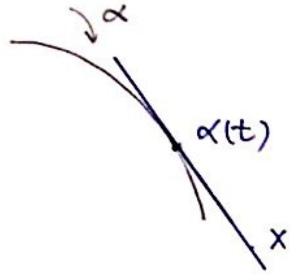
Question: · how to parametrize the tangent surface.

· if give a parametrization, we want $E = F = 1$, $G = 0$ i.e

$$I = du^2 + dv^2$$

(first fundamental form of plane)

7: ( )



We may assume t as arclength of curve α

$x(t,v) = \alpha(t) + v\,\alpha'(t)$ $\quad (t,v) \in I \times \mathbb{R}$ $\quad$ (*)

$X_t = \alpha'(t) + v\,\alpha''(t)\,, \quad X_v = \alpha'(t)$

$E(t,v) = X_t \cdot X_t = (\alpha'(t) + v\,\alpha''(t)) \cdot (\alpha'(t) + v\,\alpha''(t))$ $\qquad$ Recall $\alpha'(t) = \vec{t}(t)$, $\alpha''(t) = k(t)\vec{n}(t)$

$= |\alpha'(t)|^2 + v\,|\alpha''(t)|^2 = 1 + k^2(t)\,v^2$

$F(t,v) = X_t \cdot X_v = |\alpha'(t)|^2 = 1$

$G(t,v) = X_v \cdot X_v = 1$

$I(t,v) = (1 + k^2(t)\,v^2)\,dt^2 + 2\,dt\,dv + dv^2$ $\qquad$ (1)

Recall the first fundamental form of plane

if we use $x-y$ cordinate
the plane is parametrized by $(x, y, 0)$
then $I = dx^2 + dy^2$ $\qquad$ (2)

if we use polar coordinate
the plane is parametrized by $(r, \theta, 0)$
$r \geq 0, \quad \theta \in [0, 2\pi)$
then $I = dr^2 + r^2\,d\theta^2$ $\qquad$ (3)

One idea is to reparametrize (*) by change of variables (See P.221 - 225)

eg $\begin{cases} x = x(t,v) \\ y = y(t,v) \end{cases}$ $\qquad$ $\begin{cases} r = r(t,v) \\ \theta = \theta(t,v) \end{cases}$ $\qquad$ $\boxed{\begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{array}}$

s.t. $I(t,v) = dx^2 + dy^2$ $\qquad$ $I(t,v) = dr^2 + r^2\,d\theta^2$

It is not very easy to find such changes, Because this needs to solve some non-linear differential equations. For example, let's try to change (1) into (3)

So we need to give the relation $\begin{cases} r = r(t,v) \\ \theta = \theta(t,v) \end{cases}$

$dr = r_t\,dt + r_v\,dv$
$d\theta = \theta_t\,dt + \theta_v\,dv$

then $dr^2 + r^2 d\theta^2 = (r_t^2 + r^2\theta_t^2)\,dt^2 + 2(r_t r_v + r^2\theta_t\theta_v) + (r_v^2 + r^2\theta_v^2)\,dv^2$

so we need to solve the equations for unknowns $r_t, r_v, r, \theta_t, \theta_v, \theta$:

$$\begin{cases} r_t^2 + r^2\theta_t^2 = 1 + k^2(t)\,v^2 \\ r_t r_v + r^2\theta_t\theta_v = 1 \\ r_v^2 + r^2\theta_v^2 = 1 \end{cases}$$

← nonlinear differential equations, cannot be easily solved !

∴ The first idea doesn't work! :(

The second idea Minding's Theorem P288.

Let's try to compute the Gaussian curvature of $X(t,v)$, if it is zero, then it is locally isometric to plane.

Thm (Minding) P288	Any two regular surfaces with the same CONSTANT Gaussian curvature are locally isometric.

Rmk 1. The proof of Minding's Thm uses the existence of geodesic polar coordinates. The existence (of such coordinates) itself is equivalent to the existence of solutions for some nonlinear differential equations.

Rmk 2. If two regular surfaces satisfy $K_{S_1} = K_{S_2} \neq$ constant, then S_1 and S_2 might not be isometric.

Rmk 3. This deep theorem will give you some feelings about Gaussian curvature.

Go back to the computation of K for $X(t,v) = \alpha(t) + v\alpha'(t)$

(t - arclength of $\alpha(t)$)

$X_t = \alpha'(t) + v\alpha''(t) = t(t) + v\,k(t)\,n(t)$

$X_v = t(t)$ $\quad X_{tt} = (k(t) + v\,k'(t))\,n(t) - v\,k^2(t)\,t(t) + v\,k(t)\,\tau(t)\,b(t)$

$X_{tv} = k(t)\,n(t)$ $\quad X_{vv} = 0$ $\quad X_t \times X_v = v\,k(t)\,b(t)$ $\quad N = (\operatorname{sign} v)\,b(t)$

(Recall $k(t) > 0$)

$e = \langle N, X_{tt}\rangle = |v|\,k(t)\,\tau(t)$

$f = \langle N, X_{tv}\rangle = 0$

$g = \langle N, X_{vv}\rangle = 0$

$K = \dfrac{eg - f^2}{EG - F^2} = 0$! Then by minding's thm, $X(t,v)$ is locally isometric to plane □

Rmk: There is a third idea as hinted in textbook

Construct a plane curve with curvature = curvature of $\alpha(t)$, then apply exercise 5. (P228) (t - as arclength)

the third proof:

Actually, exercise 5 can be easily obtained from formula ① on page ②
Now if $\alpha(t)$ is given, then $k(t)$ is determined (t - arclength)
Then by the fundamental theorem of curve, $\exists$ plane curve $\tilde{\alpha}(t)$ st. $k_{\tilde{\alpha}(t)} = k_{\alpha(t)}$
$\tilde{X}(t,v) := \tilde{\alpha}(t) + v\tilde{\alpha}'(t)$ ← obvious a plane! & $X(t,v)$ is locally isometric to plane

P229

#9. Let S_1, S_2 and S_3 be regular surfaces. Prove that

a. If $\varphi : S_1 \to S_2$ is an isometry, then $\varphi^{-1} : S_2 \to S_2$ is an isometry

b. If $\varphi : S_1 \to S_2$, $\psi : S_2 \to S_3$ are isometries, then $\psi \circ \varphi : S_1 \to S_3$ is an isometry.

Rmk. If we define the set $Iso := \{ \text{isometries of a regular surface } S \}$

$= \{ \varphi : S \to S \mid \varphi \text{ isometry} \}$

Then we know

i) $id \in Iso$

ii) If $\varphi \in Iso$, then by a) above, $\varphi^{-1} \in Iso$, and $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = id$.

iii) If $\varphi, \psi \in Iso$, then $\psi \circ \varphi \in Iso$ by b).

So we can define a (Non-abelian) GROUP structure on the SET Iso

$\forall \ \varphi, \psi \in Iso$, define $\varphi \cdot \psi := \psi \circ \varphi$.

then by above i), ii) & iii) $\Longrightarrow$ $(Iso, \cdot)$ is a group.

Since $\varphi \cdot \psi \neq \psi \cdot \varphi$ in general, it is a non-abelian (non-commutative) group.

Pf of #9). Nothing but definition

$\varphi : S_1 \to S_2$ isometry $\overset{\text{definition}}{\Longleftrightarrow}$ 1° $\varphi : S \to \bar{S}$ diffeomorphism

2° $\forall p \in S$, $\forall \ w_1, w_2 \in T_p S \Rightarrow$

$\langle w_1, w_2 \rangle_p = \langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle_{\varphi(p)}$.

Now since φ is a diffeomorphism, φ^{-1} exists, and $d(\varphi^{-1}) = (d\varphi)^{-1} : T_q S_2 \to T_{\varphi^{-1}(q)} S_1$

$\forall q \in S_2$, let $p = \varphi^{-1}(q) \in S_1$, then for any $v_1, v_2 \in T_q S_2$

$$\langle v_1, v_2 \rangle_q = \langle d(\varphi^{-1}_q)(v_1), d(\varphi^{-1}_q)(v_2) \rangle_{\varphi^{-1}(q)} .$$

The chain rule of differential will give the proof of b).

#10 (P_{229}). Let S be a surface of revolution. Prove that the rotations about its axis are isometries of S.

Rmk: This gives us an example of isometry group.

Pf. Let's parametrize surface S by $X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$

(P_{76-77})

$0 < u < 2\pi$
$a < v < b$
$f(v) > 0$

Recall, u is the angle. (see P_{77} for a picture)

After rotation by some angle u_0, S will be parametrize by

$$\tilde{X}(u,v) = (f(v)\cos(u+u_0),\ f(v)\sin(u+u_0),\ g(v))$$

By computation (P_{161}), $E = f(v)^2$, $F = 0$, $G = (f'(v))^2 + (g'(v))^2$

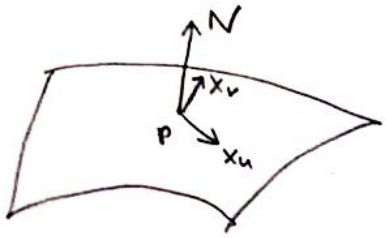
then the rotation $\varphi : S \to S$

$$X(u,v) \mapsto X(u+u_0, v) = \tilde{X}(u,v)$$

is an isometry of S.

Rmk: If S is without further symmetry, (for example, not like a sphere, which has symmetry in other directions), i.e. we assume that each isometry is an rotation, then the isometry group $= S^1 \cong (\{e^{i\theta}\}$, with group structure $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)})$

$\square$



We have a nature basis $\{X_u, X_v, N\}$

where X_u, X_v in the tangent plane of p,

$N \perp X_u, \quad N \perp X_v$ ①

X_u & X_v may not be orthogonal. (orthogonal $\Leftrightarrow F=0$)

We want to write everything ($X_{uu}, X_{uv}, X_{vv}, N_u, N_v$) interms of this basis, with some coefficients — called Christoffel symbols. Using relations ① and $X_{uv} = X_{vu}$, we will find some relations:

Now if $F=0$, then (P_{232} (2)) [See P183 Cor. 2. Locally, we can always parametrize surface by orthogonal parametrization]

$$\begin{aligned}
\Gamma_{11}^1 E &= \tfrac{1}{2}E_u & \Gamma_{11}^1 &= \tfrac{1}{2}\frac{E_u}{E} = \tfrac{1}{2}\frac{\partial(\ln E)}{\partial u}\\
\Gamma_{11}^2 G &= -\tfrac{1}{2}E_v & \Gamma_{11}^2 &= \tfrac{-1}{2}\frac{E_v}{G}\\
\Gamma_{12}^1 E &= \tfrac{1}{2}E_v \quad\Rightarrow & \Gamma_{12}^1 &= \tfrac{1}{2}\frac{E_v}{E} = \tfrac{1}{2}\frac{\partial(\ln E)}{\partial v} = \Gamma_{21}^1\\
\Gamma_{12}^2 G &= \tfrac{1}{2}G_u & \Gamma_{12}^2 &= \tfrac{1}{2}\frac{G_u}{G} = \tfrac{1}{2}\frac{\partial \ln(G)}{\partial u} = \Gamma_{21}^2\\
\Gamma_{22}^1 E &= -\tfrac{1}{2}G_u & \Gamma_{22}^1 &= \tfrac{-1}{2}\frac{G_u}{E}\\
\Gamma_{22}^2 G &= \tfrac{1}{2}G_v & \Gamma_{22}^2 &= \tfrac{1}{2}\frac{G_v}{G} = \tfrac{1}{2}\frac{\partial(\ln G)}{\partial v}
\end{aligned}$$

#1. Show that if $F=0$, then

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} \qquad \circledast$$

Pf: We know (P_{234} (5))

$$(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2 = -EK \qquad (5)$$

$$\text{LHS of (5)} = \left(\frac{1}{2}\frac{G_u}{G}\right)_u - \left(-\frac{1}{2}\frac{E_v}{G}\right)_v + \frac{1}{2}\frac{E_v}{E}\left(-\frac{1}{2}\frac{E_v}{G}\right) + \frac{1}{4}\left(\frac{G_u}{G}\right)^2 + \frac{1}{2}\frac{E_v}{G}\,\frac{1}{2}\frac{G_v}{G} - \frac{1}{2}\frac{E_u}{E}\,\frac{1}{2}\frac{G_u}{G}$$

$$= \frac{1}{2}\underset{(3)}{\frac{G_{uu}}{G}} - \frac{1}{2}\underset{(4)}{\frac{G_u^2}{G^2}} + \frac{1}{2}\underset{(1)}{\frac{E_{vv}}{G}} - \frac{1}{2}\underset{(2)}{\frac{E_vG_v}{G^2}} - \frac{1}{4}\underset{(2)}{\frac{E_v^2}{EG}} + \frac{1}{4}\underset{(4)}{\frac{G_u^2}{G^2}} + \frac{1}{4}\underset{(2)}{\frac{E_vG_v}{G^2}} - \frac{1}{4}\underset{(4)}{\frac{E_uG_u}{EG}}$$

$$\text{RHS of } \circledast = -\frac{1}{2\sqrt{EG}}\left\{\frac{E_{vv}}{\sqrt{EG}} - \frac{\frac{1}{2}E_v(EG)_v}{\sqrt{EG}\,EG} + \frac{G_{uu}}{\sqrt{EG}} - \frac{\frac{1}{2}G_u(EG)_u}{\sqrt{EG}\,EG}\right\}$$

$$\text{So } -E\cdot(\text{RHS of }\circledast) = \frac{E}{2}\left\{\underset{(1)}{\frac{E_{vv}}{EG}} - \frac{1}{2}\underset{(2)}{\frac{E_v(E_vG+EG_v)}{(EG)^2}} + \underset{(3)}{\frac{G_{uu}}{EG}} - \underset{(4)}{\frac{1\,G_u(E_uG+EG_u)}{2(EG)^2}}\right\}$$

By this careful computation, we obtain

$$-EK \overset{(5)}{=} \text{LHS of } (5) = -E\,(\text{RHS of } ⊛)$$

i.e $k = $ RHS of ⊛ , i.e ⊛ holds !

By using #1, we can show #2.

#2: Show that if $X(u,v)$ is an isothermal parametrization, that is

$$E = G = \lambda(u,v), \quad F = 0$$

then $K = -\frac{1}{2\lambda}\,\Delta(\log\lambda)$ (☆)

where $\Delta\varphi := \frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}$, the Laplacian of the function φ.

In particular, when $E = G = (u^2+v^2+c)^{-2}$ and $F = 0$, then $k = \text{const.} = 4c$

Pf. Just easy computation. put $E = G = \lambda(u,v)$ into #(1)

$$k = -\frac{1}{2\sqrt{\lambda\lambda}}\left\{\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right\}$$

Only need to show: $\Delta(\log\lambda) = \left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u$

$$\Delta(\log\lambda(u,v)) = \frac{\partial^2}{\partial u^2}(\log\lambda(u,v)) + \frac{\partial^2}{\partial v^2}(\log\lambda(u,v))$$

$$= \frac{\partial}{\partial u}\left(\frac{\lambda_u}{\lambda}\right) + \frac{\partial}{\partial v}\left(\frac{\lambda_v}{\lambda}\right)$$

$$= \left(\frac{\lambda_u}{\lambda}\right)_u + \left(\frac{\lambda_v}{\lambda}\right)_v$$

$\Rightarrow$ (☆) holds !

Now take $\lambda(u,v) = \frac{1}{(u^2+v^2+c)^2}$ you can check $k = 4c$

#7 Does there exist a surface $X = X(u, v)$ with $E = 1$, $F = 0$, $G = \cos^2 u$

$e = \cos^2 u$, $f = 0$, $g = 1$?

Idea: check the compatibility equations of surfaces. $P_{235 \sim 236}$

when $F = 0 = f$, Mainardi - Codazzi equations take the form:

(7) $e_v = \frac{E_v}{2} \left(\frac{e}{E} + \frac{g}{G} \right)$

P_{236}

(7a) $g_u = \frac{G_u}{2} \left(\frac{e}{E} + \frac{g}{G} \right)$

Now $e_v = 0$, $E_v = 0$ (7) holds.

$g_u = 0$, $\frac{G_u}{2} \left(\frac{e}{E} + \frac{g}{G} \right) = \frac{-2\cos u \cdot \sin u}{2} \left(\frac{\cos^2 u}{1} + \frac{1}{\cos^2 u} \right) \neq 0$

$\Rightarrow$ (7a) NOT holds !

Therefore, $\nexists$ such surface.

Rmk:

Let's recall the fundamental thm for curve:

give $k(s) > 0$, $\tau(s)$. both smooth function.

then $\exists$ curve C s.t curvature of C = $k(s)$

torsion of C = $\tau(s)$.

But for surface, given $\text{I} = E\, du^2 + 2F\, du\, dv + G\, dv^2$

with $EG - F^2 > 0$

& $\text{II} = e\, du^2 + 2f\, du dv + f\, dv^2$

(E, F, G, e, f, g smooth)

There are some compatibility equations:

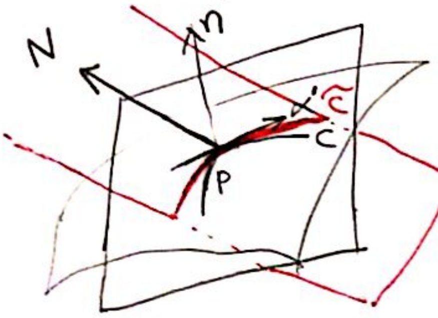
If I & II satisfy such equations, then $\exists$ surface with first (resp. second) fundamental form = I (resp. II)

If I & II are not compatible, then no such surface.

P260 #2 Prove that a curve $C \subset S$ is both an asymptotic curve and a geodesic iff C is a (segment of a) straight line

Recall:

(P148 Def. 9) An asymptotic direction of S at p is a direction of $T_p(S)$ for which the normal curvature is zero



Curvature of $\tilde{c}$ at p = normal curvature of S at p along α'

$\tilde{c}$ & C are tangent at p.

(P246 Def 8a) A regular connected curve C in S is said to be a geodesic if for every $p \in C$, the parametrization $\alpha(s)$ of a coordinate nbhd of p by the arc-length s is a parametrized geodesic; that is $\alpha'(s)$ is a parallel vector field along $\alpha(s)$.

(see Rmk below Def. 8a, & Prof Li's notes. P62) Def 8a $\Leftrightarrow$ α'' is perpendicular to the tangent plane of S at $\alpha(s)$. $\Leftrightarrow$ the normal of a curve $\alpha(s)$ is parallel to the normal of the surface at the same point.

$N = \pm n$ ← normal of C at p $\Leftrightarrow$ geodesic

$\uparrow$
normal of S at p

In above picture, for a geodesic curve C, $\boxed{\pm n = N}$ (so C & $\tilde{c}$ coincide) Curvature of $C \underset{\substack{\uparrow \\ \text{geodesic}}}{=}$ (curvature of $\tilde{c}$) $\underset{\substack{\uparrow \\ \text{asymptotic, normal curvature is zero !}}}{=} 0$

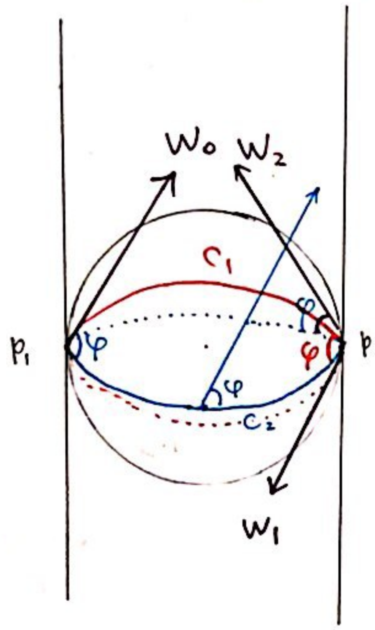
$\Rightarrow C$ is (segment of a) straight line.

"$\Leftarrow$" obvious. $\square$

You can also use : $k^2 = k_g^2 + k_n^2$, now $k_g = 0$ (geodesic)

$\Rightarrow k = k_n \underset{\text{asymptotic}}{\leftarrow} 0$

#9 Consider two meridians of a unit sphere (denote the sphere by S_1). C_1 and C_2 which make an angle φ at the point p_1. Take the parallel transport of the tangent vector w_0 of C_1, along C_1 and C_2, from the initial point p_1 to the point p_2 where the two meridians meet again, obtaining, respectively, w_1 and w_2.



Question: Compute the angle from w_1 to w_2

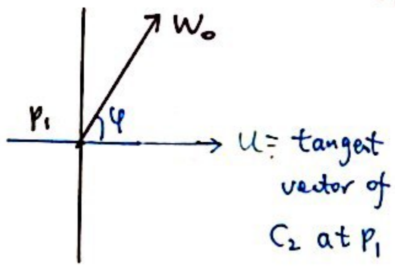
- how to geometrically describe parallel transport, in particular in the sphere ?

eg: (P241) the tangent vector field of a meridian (parametrized by arc length) of a unit sphere S^2 is a parallel field on S^2 (Fig. 4-11) (P242).

- TRICK: (P240) when two surfaces are tangent along a parametrized curve α, the covariant derivative of a field W along α is the same for both surfaces.

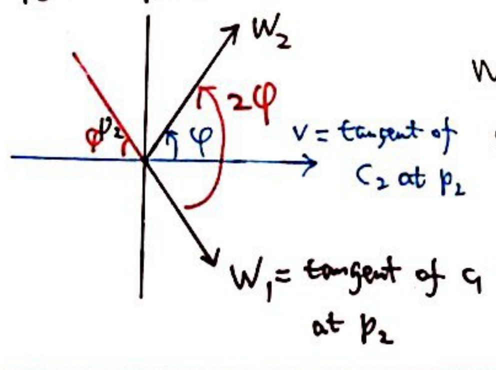
(P241) In particular, if one of the surface is plane (cone, cylinder, ie $K=0$) then the notion of parallel field along a parametrized curve reduces to that of a constant field along the curve. (Fig 4-10).

$T_{p_1}S_1 = T_{p_1}S_2$



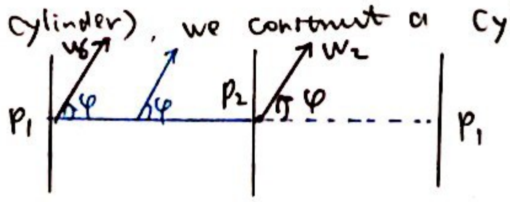
- In above picture, since both C_1 & C_2 are great circle on S^2, they both are geodesic. We draw C_2 as "equator". We draw a cylinder S_2 along C_2, i.e. the cylinder S_2 and the sphere S_1 are tangent along C_2.

$T_{p_2}S_1 = T_{p_2}S_2$



Rmk ① Since W_0 is the tangent of C_1 at P_1, The parallel transport of W_0 along C_1 is exactly the same as Fig 4-11 P242. The we got W_1 at P_2.

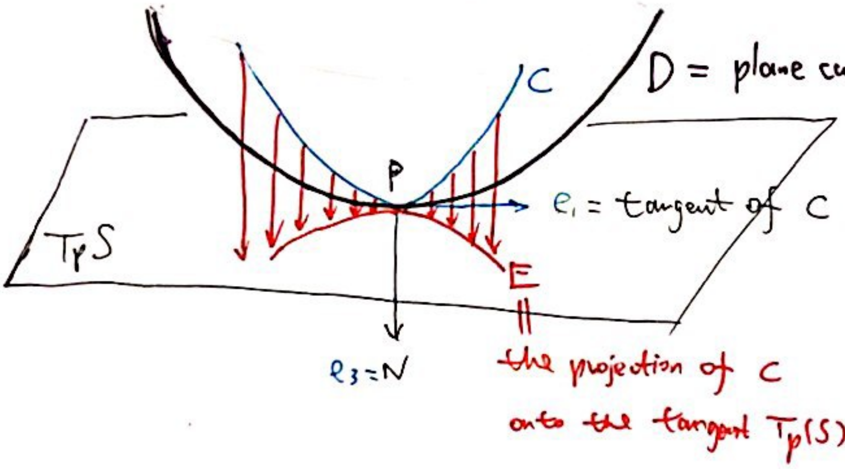
② we use the above trick (the same trick as example 1 on P243, where we construct a cylinder), we construct a cylinder. Then think the parallel transport on cylinder.



We cut the cylinder along P_1 and make it as a plane !

Angle from W_1 to $W_2 = 2\varphi$ □

P361 #10 Show that the geodesic curvature of an oriented curve $C \subset S$ at a point $p \in C$ is equal to the curvature of the plane curve by projecting C onto the tangent plane $T_p(S)$ along the normal to the surface at p.



D = plane curve by cutting the surface by the plane spanned by N & V.

curvature of D, $k_D = k_n$ normal curvature. (P142 Meusnier)

curvature of C, $k_C = k$

$k^2 = k_g^2 + k_n^2$ (P249)

We need to show: curvature of E, $k_E = k_g$

i.e. $k_C(p)^2 = k_E(p)^2 + k_D(p)^2$ where $k_C(p)/k_E(p)/k_D(p)$ means curvature of curve C/E/D

Rmk: This will give us a geometric intuition on k_n and k_g AT THE POINT P!

Let C be parametrized by arclength s : C : $\alpha(s)$, $\alpha(0) = p$

$$\left.\begin{aligned} e_1 &:= \alpha'(0) \\ e_3 &:= N \\ e_2 &:= N \times \alpha'(0) \end{aligned}\right\} \Rightarrow \text{ then } e_1, e_2, e_3 \text{ right-hand orthornormal frame}$$

Recall $\dfrac{D\alpha'(s)}{ds} = k_g\,(N \times \alpha'(s))$ i.e. $\dfrac{De_1}{ds} = k_g\, e_2$ (definition of k_g)

$\dfrac{D\alpha'(s)}{ds} = \dfrac{d\alpha'(s)}{ds} - \langle \dfrac{d\alpha'(s)}{ds}, \vec{N}\rangle \vec{N}$ i.e. $\dfrac{De_1}{ds} = \dfrac{de_1}{ds} - k_n e_3$, $\boxed{\alpha''(s) = \frac{de_1}{ds} = k_g e_2 + k_n e_3}$

$k_D(p) = \langle \dfrac{d\alpha'(s)}{ds}, N\rangle\Big|_{s=0} = \langle \dfrac{de_1}{ds}, e_3\rangle = \langle \dfrac{De_1}{ds} + k_n e_3, e_3\rangle = \langle k_g e_2 + k_n e_3, e_3\rangle = k_n$

$k_E(p) \underset{?}{=} \langle \dfrac{d\alpha'(s)}{ds}, e_2\rangle\Big|_{s=0} = \langle \dfrac{de_1}{ds}, e_2\rangle = \langle \dfrac{De_1}{ds}, e_2\rangle = k_g$

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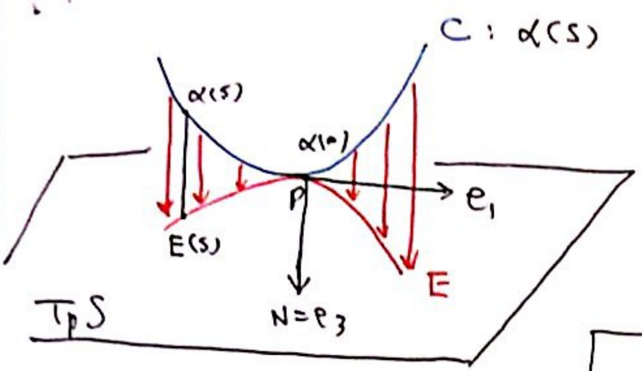
i.e. $k_E(p) = \langle \alpha''(0), e_2\rangle = k\langle n, e_2\rangle$ □



Let C be parametrized by $\alpha(s)$

$\alpha(0) = p$

The curve E can be parametrized by

$$\boxed{\begin{array}{l} E(s) - E(0) = \big(\alpha(s) - \alpha(0)\big) - \langle \alpha(s) - \alpha(0), N \rangle N \\ \text{here } E(0) = \alpha(0) = p. \quad s\text{ — arclength of } \alpha \text{ but might not be arclength of } E(s). \end{array}} \quad ☆$$

$\alpha'(0) = e_1$

$\alpha''(0) = kn$

Since s may not be the arclength of curve $E = E(s)$, the curvature of E at point p is given by the formula

$$k_E(p) = \lim_{s \to 0} \frac{|E'(s) \times E''(s)|}{|E'(s)|^3} = \frac{|E'(0) \times E''(0)|}{|E'(0)|^3}$$

From ☆

$$E'(s) = \alpha'(s) - \langle \alpha'(s), N \rangle N$$

$$\left(E'(0) = \alpha'(0) - \langle \alpha'(0), N \rangle N = \alpha'(0) \right)$$

then

$$E''(s) = \alpha''(s) - \langle \alpha''(s), N \rangle N$$

(N is the normal of surface S at p, so N is independent of s, i.e. $\frac{dN}{ds} = 0$)

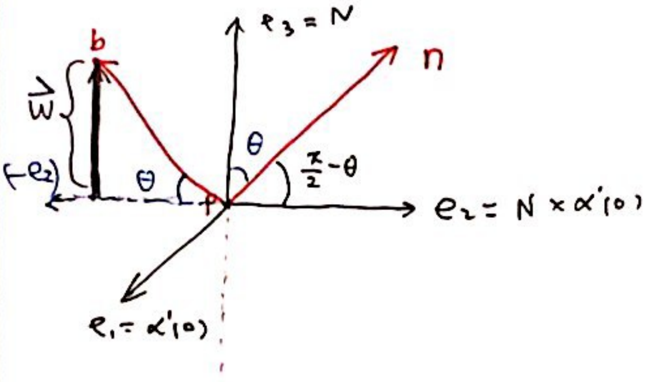
$$E'(s) \times E''(s) = \alpha'(s) \times \alpha''(s) - \langle \alpha''(s), N \rangle\, \alpha'(s) \times N - \langle \alpha'(s), N \rangle\, N \times \alpha''(s)$$

take $s \to 0$, & notice that $\langle \alpha'(0), N \rangle = 0$, then we obtain

$$E'(0) \times E''(0) = \alpha'(0) \times \alpha''(0) - \langle \alpha''(0), N \rangle\, \alpha'(0) \times N = kb - k\langle n, N \rangle (-e_2)$$

$$= k\big(b - \cos\theta(-e_2)\big)$$

$$= k\vec{W}$$



Let $\theta = \langle n, N \rangle$

where $|\vec{W}| = \sin\theta$

so
$$k_E(p) = \frac{|E'(0) \times E''(0)|}{|E'(0)|^3} = \frac{k \sin\theta}{|\alpha'(0)|^3}$$

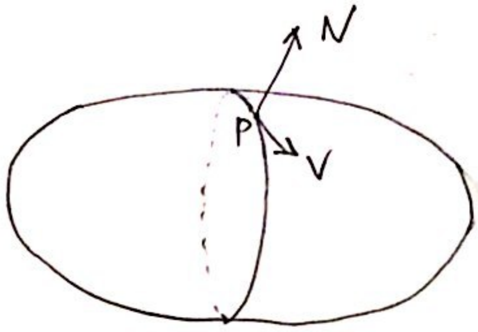
$$= k\frac{\sin\theta}{1^3} = k\sin\theta = k\cos\left(\tfrac{\pi}{2} - \theta\right)$$

$$= k\langle n, e_2 \rangle$$

□

Euler's idea on studying surfaces : at a point $p \in S$.

cutting a watermelon (surface) through normal direction N along diretion V.

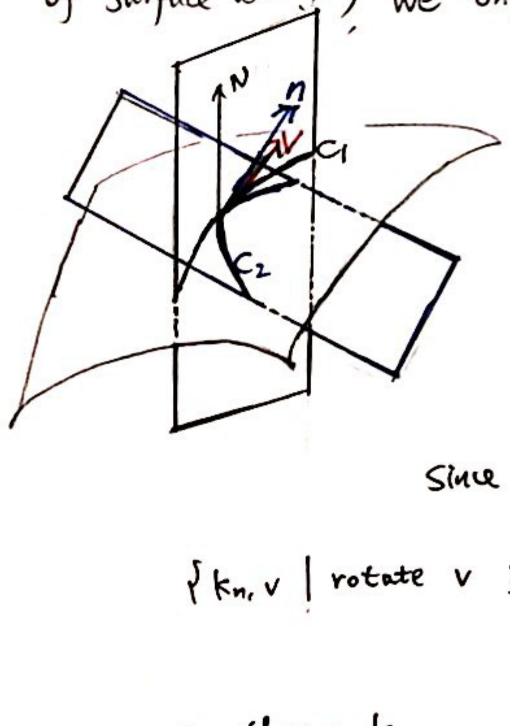


We can rotate V in the tangent plane of surface at P. Then we will get a family of plane curves.

The information of all those plane curves = information of surface at P.

curvature of plane curve (in diretion V) = normal curvature in diretion V.

Meusnier tells us : if we have a diretion v (in the tangent plane of surface at p) we only need to cut through N



k_n on diretion v = curvature of C_1

= curvature of $C_2 \cdot \cos\langle n, N\rangle$

rotate v along N, we get a family of k_n

Since k_n depends on v, we write k_n as $k_{n,v}$.

$\{k_{n,v} \mid \text{rotate } v\ 360^\circ\}$ — continuous function on a compact set

— must obtain its max'l & min'l.

max'l $:= k_1$, min'l $:= k_2$ call them principal curvature,

$\rightsquigarrow$ principal diretion e_1, e_2. (Here we need the help of Gauss : Gauss map).

Euler formula : $k_{n,v} = k_1 \cos^2\theta + k_2 \sin^2\theta$ $\quad \theta = \text{angle}\,\langle \overset{\frown}{e_1, v}\rangle$

· Gauss's idea on studing surface

study dN $\overset{\text{linear algebra}}{\rightsquigarrow}$

$dN(e_i) = -k_i e_i$

eigen vector e_1, e_2 with eigen value $-k_1$, $-k_2$.

if $\underbrace{k_1 \neq k_2}_{\text{both nonzero}}$: $\langle e_1, e_2 \rangle = \langle -\frac{1}{k_1} dN(e_1), e_2 \rangle$

$= -\frac{1}{k_1} \langle e_1, dN(e_2) \rangle$

$= \frac{k_2}{k_1} \langle e_1, e_2 \rangle$

$\Rightarrow \underbrace{(\frac{k_2}{k_1} - 1)}_{\neq 0} \langle e_1, e_2 \rangle = 0 \Rightarrow \langle e_1, e_2 \rangle = 0$

$e_1 \perp e_2$

$K := k_1 k_2$, $\quad H := \frac{1}{2}(k_1 + k_2)$.

Computation way

$$K = \frac{eg - f^2}{EG - F^2} \qquad H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}$$

Another geometric meaning of K (P167)

$$K(p) = \lim_{A \to 0} \frac{A'}{A}$$

You will see the third geometric meaning of K : Theorema Egregium P234

Crucial Observation for Gauss map $N: S \longrightarrow S^2$

① we can identify $\boxed{T_p S \equiv T_{N(p)} S^2}$

then $dN_p : T_p S \to T_{N(p)} S^2 \equiv T_p S$, i.e

dN_p is an endomorphism on $T_p S$

② $\boxed{\langle N, X_u \rangle = 0, \quad \langle N, X_v \rangle = 0}$ ⇐ Obvious but crucial

$\langle N_v, X_u \rangle + \langle N, X_{uv} \rangle = 0 \qquad \langle N_u, X_v \rangle + \langle N, X_{vu} \rangle = 0$

$X_{uv} = X_{vu}$

$\Rightarrow \langle N_u, X_v \rangle = \langle X_u, N_v \rangle \Rightarrow dN_p$ self adjoint (P140-141)

Riemann's idea on studying surfaces

Gauss (1777-1855) gave his "Theorema egregium" (Latin: "Remarkable Theorem") in 1827: The Gaussian curvature of a surface is invariant under local isometry.

His student Riemann (1826-1866) gave the new idea:

The surfaces can be studied by themselves, without embedding them in $\mathbb{R}^3$!

Riemann delivered his probationary lecture as a candidate for an unpaid lectureship at Göttingen in 1854: "On the Hypotheses which lie at the Basis of Geometry". (You can download this paper as the link on my webpage)

He gave the ideas:

1) study surfaces themselves, i.e intrinsic geometry, or "the geometry of first fundamental form".

2) initial concept of manifolds —— locally like n-dim Euclidean space.

3) propose to distinguish the metric properties from the topology properties.
   He defined metric structures on surfaces —— Now called Riemannian manifolds.
   $\sum_{i,j} g_{ij}(p)\, dx^i dx^j$

4) importance of infinite dimensional space
   eg: the set of all real-valued functions on a space

Riemann's idea and geometry are just the mathematical fundations of Einstein's General Relativity Theory (1915)

For the interesting, amazing stories, you can check wikipedia and the following book:

M. Spivak, A comprehensive introduction to differential geometry, Vol. 2,
QA641. S59 1979 v.2 in our library!