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AN EXACT INEQUALITY INVOLVING LUXEMBURG NORM AND CONJUGATE-ORLICZ NORM IN $L^{p(x)}(\Omega)$

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Let $L^{p(x)}(\Omega)$ be the variable exponent Lebesgue space where $p:\Omega\to[1,\infty]$. Denote by $\|\cdot\|_{p(x)}$ and $\|\cdot\|^{o}_{p(x)}$ the Luxemburg norm and the conjugate-Orlicz norm in $L^{p(x)}(\Omega)$ respectively. The authors prove an exact inequality $\|\cdot\|_{p(x)}\le\|\cdot\|^{o}_{p(x)}\le d_{(p_-,\,p_+)}\|\cdot\|_{p(x)}$ where $d_{(p_-,\,p_+)}$ is a constant depending on $p_-=\operatorname{ess\,inf}_\Omega p(x)$ and $p_+=\operatorname{ess\,sup}_\Omega p(x)$. When $1<p_-<p_+<\infty$,

$$d_{(p_-,\,p_+)}=\Big(\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\Big)^{\frac{p_+-1}{p_+-p_-}}\Big(\frac{p_+^{p_+}}{(p_+-1)^{p_+-1}}\Big)^{\frac{p_--1}{p_+-p_-}}+\Big(\frac{p_-^{p_-}}{(p_--1)^{p_--1}}\frac{(p_+-1)^{p_+-1}}{p_+^{p_+}}\Big)^{\frac{1}{p_+-p_-}},$$

when $p_-=1$ or $p_+=\infty$, $d_{(p_-,\,p_+)}$ is the corresponding limit.

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§1. INTRODUCTION

Let (Ω, Σ, μ) be a complete σ-finite measure space and $p : \Omega \to [1, \infty]$ a measurable function. There are many researches about the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ (see [1–4, 6–9]). By $S(\Omega)$ we denote the set of all measurable functions $u : \Omega \to \mathbb{R}$. For convenience, we introduce the notations

$$\Omega_1 = \{x \in \Omega : p(x) = 1\}, \ \Omega_+ = \{x \in \Omega : p(x) \in (1, \infty)\}, \ \Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

It is usually to define the space $L^{p(x)}(\Omega)$ and norm $\|\cdot\|_{\rho_p}$ by introducing a convex modular ρ_p on $S(\Omega)$ (see [1,6,9]), that is

$$\rho_p(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} d\mu + \operatorname*{ess\,sup}_{\Omega_\infty} |u(x)|, \tag{1.1}$$

$$L^{p(x)}(\Omega) = \{u \in S(\Omega) : \ \exists \lambda > 0 \text{ such that } \rho_p(u/\lambda) < \infty\}, \tag{1.2}$$

$$\|u\|_{\rho_p} = \inf\{\lambda > 0 : \ \rho_p(u/\lambda) \le 1\}. \tag{1.3}$$

Then $(L^{p(x)}(\Omega), \ \|\cdot\|_{\rho_p})$ is a Banach space. For $D \subset \Omega$, we denote

$$p_-(D) = \operatorname*{ess\,inf}_D p(x), \quad p_+(D) = \operatorname*{ess\,sup}_D p(x).$$

Use p_- and p_+ instead of $p_-(\Omega)$ and $p_+(\Omega)$ respectively. Denote by $p^o(x)$ the conjugate number to $p(x)$, that is

$$p^o(x) = \begin{cases} \dfrac{p(x)}{p(x)-1}, & x \in \Omega_+, \\ \infty, & x \in \Omega_1, \\ 1, & x \in \Omega_\infty. \end{cases} \tag{1.4}$$

Kováčik and Rákosník [6] have introduced an equivalent norm $\|\cdot\|^o_{\rho_p}$:

$$\|u\|^o_{\rho_p} = \sup\Big\{\Big|\int_\Omega u(x)v(x)dx\Big| : \ v \in L^{p^o(x)}(\Omega), \ \rho_{p^o}(v) \le 1\Big\},$$

and proved the following inequality (see [6, Theorem 2.3])

$$c^{-1}_{p(x)}\|u\|_{\rho_p} \le \|u\|^o_{\rho_p} \le r_{p(x)}\|u\|_{\rho_p}, \tag{1.5}$$

where

$$c_{p(x)} = \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_+}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty,$$

$$r_{p(x)} = c_{p(x)} + \frac{1}{p_-(\Omega_+)} - \frac{1}{p_+(\Omega_+)}$$

with the convention $\frac{1}{\infty} = 0$. Especially when $\Omega_1 = \Omega_\infty = \emptyset$, we have $c_{p(x)} = 1$ and $r_{p(x)} = 1 + \frac{1}{p_-} - \frac{1}{p_+}$. In this case the inequality (1.5) comes

$$\|u\|_{\rho_p} \le \|u\|^o_{\rho_p} \le \Big(1 + \frac{1}{p_-} - \frac{1}{p_+}\Big)\|u\|_{\rho_p}. \tag{1.6}$$

It is easy to see that the above constant $r_{p(x)}$, as an embedding constant of $(L^{p(x)}(\Omega), \|\cdot\|_{\rho_p}) \hookrightarrow (L^{p(x)}(\Omega), \|\cdot\|^o_{\rho_p})$, is not the best. In this paper, we find the best embedding constant $d_{p(x)} = d_{(p_-,\, p_+)}$, which depends on p_- and p_+. When $1 < p_- < p_+ < \infty$,

$$d_{(p_-,\, p_+)} = \Big(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}}\Big)^{\frac{p_+ - 1}{p_+ - p_-}} \Big(\frac{p_+^{p_+}}{(p_+ - 1)^{p_+ - 1}}\Big)^{\frac{p_- - 1}{p_+ - p_-}} + \Big(\frac{p_-^{p_-}}{(p_- - 1)^{p_- - 1}} \frac{(p_+ - 1)^{p_+ - 1}}{p_+^{p_+}}\Big)^{\frac{1}{p_+ - p_-}}. \tag{1.7}$$

In this case, it is easy to verify $d_{(p_-,\, p_+)} < 1 + \frac{1}{p_-} - \frac{1}{p_+} = r_{p(x)}$ (see the Remark 3.1 below). In addition, when $\Omega_\infty \neq \emptyset$ the convex modular ρ_p defined by (1.1) is not generated by the integral of an Orlicz function $\Phi(x, \cdot)$, so the space $(L^{p(x)}(\Omega), \|\cdot\|_{\rho_p})$ is not a generalized Orlicz space in the sense of [7], but a Musielak-Nakano space. In this paper we introduce a norm $\|\cdot\|_{p(x)}$ in $L^{p(x)}(\Omega)$ such that $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a generalized Orlicz space. Let us consider a function defined by formula $\Phi(x,t) = t^{p(x)}$, $\forall t \geq 0$, $\forall x \in \Omega$, with the convention

$$t^\infty = \begin{cases} 0, & 0 \leq t \leq 1, \\ \infty, & t > 1. \end{cases} \tag{1.8}$$

Then for every $x \in \Omega$, $\Phi(x, \cdot)$ is an Orlicz function (i.e., a nonnegative lower semi-continuous convex function vanishing at zero but not identically 0 or ∞ on $(0, \infty)$). We define the Luxemburg norm in $L^{p(x)}(\Omega)$ by

$$\|u\|_{p(x)} = \inf\Big\{\lambda > 0,\ \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} d\mu \leq 1\Big\}. \tag{1.9}$$

It is easy to verify that the two norms $\|u\|_{p(x)}$ and $\|u\|_{\rho_p}$ are equivalent. Moreover when $\Omega_\infty = \emptyset$, they are same. It is also easy to see that when $\Omega = \Omega_\infty$ the norm $\|u\|_{p(x)}$ is just the same with the usual L^∞-norm $\|u\|_\infty = \operatorname{ess\,sup}_\Omega |u(x)|$. We define the conjugate-Orlicz norm $\|u\|^o_{p(x)}$ in $L^{p(x)}(\Omega)$ by

$$\|u\|^o_{p(x)} = \sup\Big\{\Big|\int_\Omega u(x)v(x)\ d\mu\Big| :\ v \in L^{p^o(x)}(\Omega),\ \int_\Omega |v(x)|^{p^o(x)} d\mu \leq 1\Big\}. \tag{1.10}$$

It is clear that $\|u\|^o_{p(x)} = \|u\|^o_{\rho_p}$ if $\Omega_1 = \Omega_\infty = \emptyset$.

When $1 < p_- < p_+ < \infty$, the constant $d_{p(x)} = d_{(p_-,\, p_+)}$ is defined by (1.7). When $p_- = 1$ or $p_+ = \infty$, $d_{p(x)}$ can be defined in the limit sense of (1.7), that is

$$d_{p(x)} = \begin{cases} d_{(p_-,\, p_+)} \quad \text{defined by (1.7)}, & \text{if } 1 < p_- < p_+ < \infty, \\ d_{(1,\, p_+)} = \lim\limits_{p_- \to 1+} d_{(p_-,\, p_+)} = 1 + \Big(\dfrac{(p_+ - 1)^{p_+ - 1}}{p_+{}^{p_+}}\Big)^{\frac{1}{p_+ - 1}}, & \text{if } 1 = p_- < p_+ < \infty, \\ d_{(p_-,\, \infty)} = \lim\limits_{p_+ \to \infty} d_{(p_-,\, p_+)} = \dfrac{(p_- - 1)^{p_- - 1}}{p_-{}^{p_-}} + 1, & \text{if } 1 < p_- < p_+ = \infty, \\ d_{(1,\, \infty)} = \lim\limits_{\substack{p_- \to 1+ \\ p_+ \to \infty}} d_{(p_-,\, p_+)} = 2. & \text{if } 1 = p_- \text{ and } p_+ = \infty, \\ \lim\limits_{q \to p_-} d_{(p_-,\, q)} = \lim\limits_{r \to p_+} d_{(r,\, p_+)} = 1, & \text{if } p_- = p_+. \end{cases} \tag{1.11}$$

The main result of this paper is the following theorem.

Theorem 1.1. The inequality

$$\|u\|_{p(x)} \le \|u\|^o_{p(x)} \le d_{p(x)}\|u\|_{p(x)}, \quad \forall\, u \in L^{p(x)}(\Omega) \tag{1.12}$$

holds and is exact. In other words we have

$$\sup_{u\in L^{p(x)}(\Omega)\setminus\{0\}} \frac{\|u\|_{p(x)}}{\|u\|^o_{p(x)}} = 1, \tag{1.13}$$

$$\sup_{u\in L^{p(x)}(\Omega)\setminus\{0\}} \frac{\|u\|^o_{p(x)}}{\|u\|_{p(x)}} = d_{p(x)}. \tag{1.14}$$

Remark 1.1. When $p_- = p_+$, that is, $p(x) \equiv p \in [1, \infty]$ a.e., $(L^{p(x)}(\Omega), \|u\|_{p(x)})$ is the usual Lebesgue space $(L^p(\Omega), \|u\|_p)$. It is well known that $\|u\|_p = \|u\|^o_p$ in this case. This is a special case of Theorem 1.1. In order to prove Theorem 1.1, we need only to deal with the case $p_- < p_+$.

Remark 1.2. In the inequality (1.5), $c_{p(x)}$ =1, 2 or 3, the constants $c_{p(x)}$ and $r_{p(x)}$ are discontinuous in (p_-, p_+). While in our inequality (1.12), $d_{p(x)} = d_{(p_-, p_+)}$ is continuous in (p_-, p_+). This reflects the rationality of using the norm $\|u\|_{p(x)}$ defined by (1.9) with the convention (1.8).

The paper is organized as follows. In Section 2 we prove the conclusion (1.13). In Section 3 we prove $\|u\|^o_{p(x)} \le d_{p(x)}\|u\|_{p(x)}$. In Section 4 we prove the exactness of $d_{p(x)}$.

§2. INEQUALITY $\|u\|_{p(x)}\le\|u\|^o_{p(x)}$ AND ITS EXACTNESS

The aim of this section is to prove (1.13). First let us prove the following lemma.

Lemma 2.1.

$$\|u\|_{p(x)} \le \|u\|^o_{p(x)}, \quad \forall\, u \in L^{p(x)}(\Omega). \tag{2.1}$$

Proof. Let $u \in L^{p(x)}(\Omega)$ and $\|u\|_{p(x)} = 1$. We can assume $u(x) \ge 0$ for $x \in \Omega$ because the norms of $u(x)$ and $|u(x)|$ are equal. Let $c = \|u\|_{L^\infty(\Omega_\infty)}$ (when $\Omega_\infty = \emptyset$, we assume $c = 0$). By the definition (1.9) of $\|u\|_{p(x)}$, $c \le 1$.

Case (i) $c < 1$.

In this case we have $\int_\Omega |u(x)|^{p(x)}d\mu = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}d\mu = 1$. Let

$$v(x) = \begin{cases} |u(x)|^{p(x)-1}, & \text{if } x \in \Omega\setminus\Omega_\infty, \\ 0, & \text{if } x \in \Omega_\infty. \end{cases}$$

Then $\int_\Omega |v(x)|^{p^o(x)}d\mu = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}d\mu = 1$. By the definition (1.10) of $\|u\|^o_{p(x)}$, we have

$$\|u\|^o_{p(x)} \ge \int_\Omega u(x)v(x)d\mu = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}d\mu = 1 = \|u\|_{p(x)}.$$

Case (ii) $c = 1$

Given any $\varepsilon > 0$, there exists $D \subset \Omega_\infty$ such that $\mu(D) > 0$ and $u(x) \ge 1 - \varepsilon$ for $x \in D$. Let

$$v(x) = \begin{cases} \frac{1}{\mu(D)}, & \text{if } x \in D, \\ 0, & \text{if } x \in \Omega\setminus D. \end{cases}$$

Then

$$\int_\Omega |v(x)|^{p^o(x)}d\mu = \int_D \frac{1}{\mu(D)}d\mu = 1.$$

We have

$$\|u\|^o_{p(x)} \geq \int_\Omega u(x)v(x)d\mu = \int_D u(x)\frac{1}{\mu(D)}d\mu \geq 1-\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\|u\|^o_{p(x)} \geq 1 = \|u\|_{p(x)}$. The proof is complete.

Let us turn to the proof of (1.13). The inequality (2.1) means

$$\sup_{u\in L^{p(x)}(\Omega)\backslash\{0\}} \frac{\|u\|_{p(x)}}{\|u\|^o_{p(x)}} \leq 1.$$

In order to prove (1.13), it suffices to prove

$$\sup_{u\in L^{p(x)}(\Omega)\backslash\{0\}} \frac{\|u\|_{p(x)}}{\|u\|^o_{p(x)}} \geq 1. \tag{2.2}$$

When $\Omega_1 \neq \emptyset$ or $\Omega_\infty \neq \emptyset$, by the Remark 1.1 it is easy to see that there exists $u \in L^{p(x)}(\Omega)$ such that $\|u\|_{p(x)} = \|u\|^o_{p(x)} = 1$. Thus (2.2) holds. We now assume $\Omega_1 = \Omega_\infty = \emptyset$, i.e. $\Omega = \Omega_+$. Given any $\varepsilon > 0$, we can find $D \subset \Omega_+$ such that $\mu(D) > 0$ and $p_+(D) - p_-(D)$ small enough satisfying

$$\frac{1}{\bar p} + \frac{1}{\bar q} \leq 1+\varepsilon, \quad \forall\, \bar p \in [\,p_-(D),\ p_+(D)\,],\ \forall\, \bar q \in [\,(p^o)_-(D),\ (p^o)_+(D)\,].$$

Take $u \in L^{p(x)}(\Omega)$ such that $\|u\|_{p(x)} = 1$ and $\operatorname{supp} u \subset D$. Then $\int_D |u(x)|^{p(x)}d\mu = 1$. For any $v \in L^{p^o(x)}(\Omega)$ with $\int_\Omega |v(x)|^{p^o(x)}d\mu \leq 1$ there holds

$$\begin{aligned}\Big|\int_\Omega u(x)v(x)d\mu\Big| &= \Big|\int_D u(x)v(x)d\mu\Big| \leq \int_D \Big(\frac{1}{p(x)}|u(x)|^{p(x)} + \frac{1}{p^o(x)}|v(x)|^{p^o(x)}\Big)d\mu\\ &= \frac{1}{\bar p}\int_D |u(x)|^{p(x)}d\mu + \frac{1}{\bar q}\int_D |v(x)|^{p^o(x)}d\mu \leq \frac{1}{\bar p} + \frac{1}{\bar q} \leq 1+\varepsilon,\end{aligned}$$

where $\bar p$ and $\bar q$ are some constants in $[\,p_-(D),\ p_+(D)\,]$ and in $[\,(p^o)_-(D),\ (p^o)_+(D)\,]$ respectively. It implies that $\|u\|^o_{p(x)} \leq 1+\varepsilon$, and thus $\frac{\|u\|_{p(x)}}{\|u\|^o_{p(x)}} \geq \frac{1}{1+\varepsilon}$. So the inequality (2.2) holds. This proves (1.13).

§3. PROOF OF INEQUALITY $\|u\|^o_{p(x)} \leq d_{p(x)}\|u\|_{p(x)}$

In this section we will prove

$$\|u\|^o_{p(x)} \leq d_{p(x)}\|u\|_{p(x)}, \quad \forall\, u \in L^{p(x)}(\Omega). \tag{3.1}$$

First, let us give some preliminaries. Denote

$$\Phi(x,t) = \frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\, t^{p(x)}, \quad \forall x\in\Omega, \quad \forall t\geq 0. \tag{3.2}$$

Here we appoint that

$$\begin{aligned} &\Phi(x,t)=t, \quad \text{if } x\in\Omega_1,\\ &\Phi(x,t)=0, \quad \text{if } x\in\Omega_\infty \text{ and } 0\le t\le 1,\\ &\Phi(x,t)=\infty, \quad \text{if } x\in\Omega_\infty \text{ and } t>1. \end{aligned}$$

Denote by $\Phi^*(x,\cdot)$ the complementary function to the function $\Phi(x,\cdot)$, that is

$$\Phi^*(x,s)=\sup_{t\ge 0}\{ts-\Phi(x,t)\}, \quad \forall\, x\in\Omega,\ \forall\, s\ge 0. \tag{3.3}$$

It is easy to see that

$$\Phi^*(x,s)=s^{p^o(x)}, \quad \forall\, x\in\Omega,\ \forall\, s\ge 0. \tag{3.4}$$

Set

$$L^\Phi(\Omega)=\Big\{u\in S(\Omega): \exists\lambda>0 \text{ such that } \int_\Omega \Phi\Big(x,\Big|\frac{u(x)}{\lambda}\Big|\Big)\, d\mu<\infty\Big\}.$$

In $L^\Phi(\Omega)$, the Luxemburg norm $\|\cdot\|_\Phi$, Orlicz norm $\|\cdot\|_\Phi^o$, Amemiya norm $\|\cdot\|_\Phi^A$ are defined (see [7]) respectively by

$$\|u\|_\Phi=\inf\{\lambda>0: \int_\Omega \Phi(x,|u(x)/\lambda|)d\mu\le 1\}, \tag{3.5}$$

$$\|u\|_\Phi^o=\sup\Big\{\Big|\int_\Omega u(x)v(x)\, d\mu\Big| : v\in L^{\Phi^*}(\Omega),\ \int_\Omega \Phi^*(x,|v(x)|)d\mu\le 1\Big\}, \tag{3.6}$$

$$\|u\|_\Phi^A=\inf\Big\{\frac{1}{k}\Big(\int_\Omega \Phi(x,|ku(x)|\Big)d\mu+1): k>0\Big\}. \tag{3.7}$$

We can get $L^{p(x)}(\Omega)\subset L^\Phi(\Omega)$ because of $\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\le 1$. For any $u\in L^{p(x)}(\Omega)$, from (3.4), (3.6) and (1.10) we know

$$\|u\|_{p(x)}^o=\|u\|_\Phi^o. \tag{3.8}$$

By [3], we know

$$\|u\|_\Phi^o=\|u\|_\Phi^A. \tag{3.9}$$

(When $\Phi(x,t)=\Phi(t)$ is a general Orlicz function, the equality (3.9) has been proved by Hudzik and Maligranda in [5]). In order to prove the inequality (3.1) it suffices to prove the inequality

$$\|u\|_\Phi^A\le d_{p(x)}\|u\|_{p(x)}, \quad \forall\, u\in L^{p(x)}(\Omega). \tag{3.10}$$

We prove (3.10) in lemma-size steps. We assume hereafter $p_-<p_+$ because of Remark 1.1.
Lemma 3.1. (3.10) holds when $1<p_-<p_+<\infty$.
Proof. Clearly $\Omega=\Omega_+$. Let $u\in L^{p(x)}(\Omega)$ and $\|u\|_{p(x)}=1$. It suffices to prove

$$\|u\|_\Phi^A\le d_{p(x)}=d_{(p_-,\,p_+)}. \tag{3.11}$$

For $k>0$ let

$$G(k)=\frac{1}{k}\Big(\int_\Omega \frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\, k^{p(x)}|u(x)|^{p(x)}\, d\mu+1\Big). \tag{3.12}$$

By the definition (3.7) of $\|u\|_\Phi^A$, we have $\|u\|_\Phi^A = \inf_{k>0} G(k)$. For every $k>0$, define $f_k : [p_-, p_+] \to \mathbb{R}$ by

$$f_k(p) = \frac{(p-1)^{p-1}}{p^p} k^p \quad \text{for} \quad p \in [p_-, p_+]. \tag{3.13}$$

We can obtain

$$\sup_{p_- \le p \le p_+} f_k(p) = \max\{f_k(p_-),\ f_k(p_+)\} = f_k(p_*), \tag{3.14}$$

where $p_* = p_-$ or $p_* = p_+$ as following. Denote

$$\bar{k} = \Big(\frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} \frac{p_+^{p_+}}{(p_+ - 1)^{p_+ - 1}}\Big)^{\frac{1}{p_+ - p_-}}. \tag{3.15}$$

Note that it follows from (3.15) that

$$\frac{p_+}{p_+ - 1} < \bar{k} < \frac{p_-}{p_- - 1}. \tag{3.16}$$

It is easy to see that, when $0 < k < \bar{k}$, we have $f_k(p_-) > f_k(p_+)$ and then $p_* = p_-$; when $k > \bar{k}$, we have $f_k(p_-) < f_k(p_+)$ and then $p_* = p_+$; when $k = \bar{k}$, we have $f_k(p_-) = f_k(p_+)$ and then $p_* = p_-$ or p_+. Thus we have

$$G(k) \le \frac{1}{k}(f_k(p_*) + 1), \tag{3.17}$$

$$\|u\|_\Phi^A = \inf_{k>0} G(k) \le \inf_{k>0} \frac{1}{k}(f_k(p_*) + 1). \tag{3.18}$$

For every fixed $p_* > 1$, consider the function of k

$$g_{p_*}(k) = \frac{1}{k}(f_k(p_*) + 1) = \frac{1}{k}\Big(\frac{(p_* - 1)^{p_* - 1}}{p_*^{p_*}} k^{p_*} + 1\Big), \quad \forall\ k > 0. \tag{3.19}$$

Denote $k^* = \frac{p_*}{p_* - 1}$. Then $g'_{p_*}(k^*) = 0$. We can obtain that $g_{p_*}(k)$ is decreasing in $(0,\ k^*]$ and increasing in $[k^*,\ \infty)$. When $k \in (0,\ \bar{k}]$,

$$p_* = p_-, \quad k^* = \frac{p_-}{p_- - 1} > \bar{k},$$

$g_{p_-}(k)$ is decreasing in $(0,\ \bar{k}]$. Thus

$$\inf_{0<k\le\bar{k}} \frac{1}{k}(f_k(p_*) + 1) = \inf_{0<k\le\bar{k}} g_{p_-}(k) = g_{p_-}(\bar{k}) = d_{(p_-,\ p_+)}. \tag{3.20}$$

When $k \in [\bar{k},\ \infty)$,

$$p_* = p_+, \quad k^* = \frac{p_+}{p_+ - 1} < \bar{k},$$

$g_{p_+}(k)$ is increasing in $[\bar{k},\ \infty)$. Thus

$$\inf_{k\ge\bar{k}} \frac{1}{k}(f_k(p_*) + 1) = \inf_{k\ge\bar{k}} g_{p_+}(k) = g_{p_+}(\bar{k}) = d_{(p_-,\ p_+)}. \tag{3.21}$$

By (3.20), (3.21) and (3.18) we obtain

$$\|u\|_\Phi^A = \inf_{k>0} G(k) \le d_{(p_-,\ p_+)}.$$

The proof is complete.

Remark 3.1. By the proof of Lemma 3.1, we can get $d_{(p_-,\ p_+)} = g_{p_-}(\bar{k}) = g_{p_+}(\bar{k})$, where $\bar{k}$ is defined by (3.15), $g_{p_*}(k)$ is defined by (3.19). Because $g_{p_-}(k)$ is decreasing in $(0,\ \bar{k}]$ and $\frac{p_+}{p_+-1} < \bar{k}$, we obtain

$$\begin{aligned}
d_{(p_-,\ p_+)} &= g_{p_-}(\bar{k}) < g_{p_-}\big(\frac{p_+}{p_+-1}\big) \\
&= 1 + \frac{1}{p_-}\Big(\frac{p_--1}{p_-}\frac{p_+}{p_+-1}\Big)^{p_--1} - \frac{1}{p_+} \\
&< 1 + \frac{1}{p_-} - \frac{1}{p_+} = r_{p(x)}.
\end{aligned}$$

Lemma 3.2. (3.10) holds when $\Omega_\infty = \emptyset$.

Proof. Actually, the proof of Lemma 3.1 is also valid for the case $1 = p_- < p_+ < \infty$. Here we only point out that when $p_- = 1$,

$$\frac{(p_- - 1)^{p_--1}}{p_-^{\ p_-}} = 1, \quad \bar{k} = \Big(\frac{p_+^{p_+}}{(p_+-1)^{p_+-1}}\Big)^{\frac{1}{p_+-1}}$$

and $d_{p(x)} = d_{(1,\ p_+)}$. When $\Omega_\infty = \emptyset$ and $p_+(\Omega_+) = \infty$, we still use the symbol as in the proof of Lemma 3.1. In this case,

$$f_k(p) = \frac{(p-1)^{p-1}}{p^p}k^p, \qquad \forall\, p \in [p_-,\ \infty).$$

We have

$$\sup_{p\in[p_-,\ \infty)} f_k(p) = \begin{cases} \infty, & \text{if } \ k > 1, \\ f_k(p_-), & \text{if } \ k \le 1. \end{cases}$$

Then for $u \in L^{p(x)}(\Omega)$ with $\|u\|_{p(x)} = 1$, we have

$$\begin{aligned}
\|u\|_\Phi^A &= \inf_{k>0} G(k) \le \inf_{k>0} \frac{1}{k}\Big(\sup_{p\in[p_-,\ \infty)} f_k(p) + 1\Big) \\
&= \inf_{0<k\le 1} \frac{1}{k}(f_k(p_-) + 1) = \inf_{0<k\le 1} g_{p_-}(k) = g_{p_-}(1) \\
&= f_1(p_-) + 1 = \frac{(p_- - 1)^{p_--1}}{p_-^{\ p_-}} + 1 = d_{(p_-,\ \infty)}.
\end{aligned}$$

The proof is complete.

Lemma 3.3. (3.10) holds when $\Omega_\infty \ne \emptyset$.

Proof. Suppose that $u \in L^{p(x)}(\Omega)$ and $\|u\|_{p(x)} = 1$. In this case, $\|u\|_{L^\infty(\Omega_\infty)} \le 1$ and

$$\int_\Omega |u(x)|^{p(x)} d\mu = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)} d\mu \le 1.$$

Moreover,

$$\begin{aligned}\|u\|_\Phi^A &= \inf_{k>0} \frac{1}{k}\Big(\int_\Omega \Phi(x,|ku(x)|)d\mu + 1\Big)\\ &\le \inf_{0<k\le 1} \frac{1}{k}\Big(\int_{\Omega\setminus\Omega_\infty} \frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\, k^{p(x)}|u(x)|^{p(x)}\, d\mu + 1\Big)\\ &\le \inf_{0<k\le 1} \frac{1}{k}\Big(\frac{(p_- -1)^{p_- -1}}{p_-^{p_-}}\, k^{p_-} + 1\Big)\\ &= \frac{(p_- -1)^{p_- -1}}{p_-{}^{p_-}} + 1 = d_{(p_-,\,\infty)}.\end{aligned} \tag{3.22}$$

The proof is complete.

By the above lemmas, (3.1) is true.

§4. THE EXACTNESS OF $d_{p(x)}$

In this section, we will prove (1.14). The inequality (3.1) means that

$$\sup_{u\in L^{p(x)}(\Omega)\setminus\{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \le d_{p(x)}.$$

To prove (1.14), it suffices to prove

$$\sup_{u\in L^{p(x)}(\Omega)\setminus\{0\}} \frac{\|u\|_{p(x)}^o}{\|u\|_{p(x)}} \ge d_{p(x)}. \tag{4.1}$$

We use the following lemmas to complete the proof.

Lemma 4.1. (4.1) holds for $1 < p_- < p_+ < \infty$.

Proof. Let $\bar{k}$ be as in (3.15). Set

$$a = \Big(\frac{p_- - 1}{p_-}\,\bar{k}\Big)^{p_-}, \quad b = \Big(\frac{p_+ - 1}{p_+}\,\bar{k}\Big)^{p_+}.$$

Thus $a < 1$ and $b > 1$. Letting

$$\lambda_1 = \frac{b-1}{b-a}, \quad \lambda_2 = \frac{1-a}{b-a},$$

we have

$$\lambda_1, \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1 a + \lambda_2 b = 1. \tag{4.2}$$

Let $k_0 > \bar{k}$ be large enough such that

$$\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\, k_0^{p(x)} > \frac{2}{\lambda_2}, \quad \forall\, x\in\Omega. \tag{4.3}$$

Given any $\varepsilon > 0$, we can find $A_- \subset \Omega$ and $A_+ \subset \Omega$ such that

$$\mu(A_-) > 0, \quad \mu(A_+) > 0, \quad A_- \cap A_+ = \emptyset,$$

$$\Big|\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\,k^{p(x)}-\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,k^{p_-}\Big|<\frac{\varepsilon}{2},\quad \forall\, x\in A_-,\quad \forall\,\frac{1}{2}\le k\le k_0, \tag{4.4}$$

$$\Big|\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\,k^{p(x)}-\frac{(p_+-1)^{p_+-1}}{p_+^{p_+}}\,k^{p_+}\Big|<\frac{\varepsilon}{2},\quad \forall\, x\in A_+,\quad \forall\,\frac{1}{2}\le k\le k_0. \tag{4.5}$$

Take $u\in L^{p(x)}(\Omega)$ such that $\operatorname{supp} u\subset(A_-\cup A_+)$ and

$$\int_{A_-}|u(x)|^{p(x)}d\mu=\lambda_1,\qquad \int_{A_+}|u(x)|^{p(x)}d\mu=\lambda_2. \tag{4.6}$$

Thus we have

$$\int_\Omega|u(x)|^{p(x)}d\mu=\lambda_1+\lambda_2=1,\quad \|u\|_{p(x)}=1.$$

From Section 3 we already know

$$\|u\|^o_{p(x)}=\inf_{k>0}G(k)=\inf_{k>0}\frac{1}{k}\Big(\int_\Omega\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\,k^{p(x)}|u(x)|^{p(x)}\,d\mu+1\Big).$$

Since $\|u\|^o_{p(x)}\le d_{(p_-,\,p_+)}\le 2$ and $G(k)>2$ for $k\in(0,\frac{1}{2})\cup(k_0,\infty)$, we have

$$\|u\|^o_{p(x)}=\inf_{\frac{1}{2}\le k\le k_0}G(k).$$

By (4.4) and (4.5) we obtain

$$\begin{aligned}
\|u\|^o_{p(x)}&=\inf_{\frac{1}{2}\le k\le k_0}\frac{1}{k}\Big(\int_\Omega\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\,k^{p(x)}|u(x)|^{p(x)}\,d\mu+1\Big)\\
&=\inf_{\frac{1}{2}\le k\le k_0}\frac{1}{k}\Big\{\int_{A_-}\Big[\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,k^{p_-}+\Big(\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\,k^{p(x)}\\
&\quad-\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,k^{p_-}\Big)\Big]|u(x)|^{p(x)}d\mu+\int_{A_+}\Big[\frac{(p_+-1)^{p_+-1}}{p_+^{p_+}}\,k^{p_+}\\
&\quad+\Big(\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\,k^{p(x)}-\frac{(p_+-1)^{p_+-1}}{p_+^{p_+}}\,k^{p_+}\Big)\Big]|u(x)|^{p(x)}d\mu+1\Big\}\\
&\ge\inf_{\frac{1}{2}\le k\le k_0}\frac{1}{k}\Big(\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,k^{p_-}\lambda_1+\frac{(p_+-1)^{p_+-1}}{p_+^{p_+}}\,k^{p_+}\lambda_2+1-\frac{\varepsilon}{2}\Big)\\
&\ge\inf_{\frac{1}{2}\le k\le k_0}\frac{1}{k}\Big(\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,k^{p_-}\lambda_1+\frac{(p_+-1)^{p_+-1}}{p_+^{p_+}}\,k^{p_+}\lambda_2+1\Big)-\varepsilon.
\end{aligned} \tag{4.7}$$

Define

$$H(k)=\frac{1}{k}\Big(\frac{(p_--1)^{(p_--1)}}{p_-^{p_-}}\,k^{p_-}\lambda_1+\frac{(p_+-1)^{(p_+-1)}}{p_+^{p_+}}\,k^{p_+}\lambda_2+1\Big),\quad \forall\, k\in[\frac{1}{2},\ k_0].$$

It is easy to verify that $\bar k$ is the minimum point of $H(k)$ in $[\frac{1}{2},\ k_0]$ and

$$\begin{aligned}
&\inf_{\frac{1}{2}\le k\le k_0}H(k)=H(\bar k)\\
&=\lambda_1\frac{1}{\bar k}\Big(\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,\bar k^{p_-}+1\Big)+\lambda_2\frac{1}{\bar k}\Big(\frac{(p_+-1)^{p_+-1}}{p_+^{p_+}}\,\bar k^{p_+}+1\Big)\\
&=\lambda_1 d_{(p_-,\,p_+)}+\lambda_2 d_{(p_-,\,p_+)}=d_{(p_-,\,p_+)}.
\end{aligned} \tag{4.8}$$

By (4.7) and (4.8) we obtain

$$\|u\|^o_{p(x)} \ge \inf_{\frac{1}{2}\le k\le k_0} H(k) - \varepsilon \ge d_{(p_-,\, p_+)} - \varepsilon.$$

It means

$$\sup_{u\in L^{p(x)}(\Omega)\backslash\{0\}} \frac{\|u\|^o_{p(x)}}{\|u\|_{p(x)}} \ge d_{(p_-,\, p_+)} - \varepsilon.$$

Thus (4.1) holds since $\varepsilon > 0$ is arbitrary.

Lemma 4.2. (4.1) holds for $1 = p_- < p_+ < \infty$.

Proof. The proof is similar to that of Lemma 4.1. Please refer to the proof of Lemma 3.2 and notice that $a = 0$.

Lemma 4.3 (4.1) holds when $\Omega_\infty = \emptyset$ and $p_+(\Omega_+) = \infty$.

Proof. $\forall\ \varepsilon > 0$, for every natural number n, denote

$$\Omega^{(n)} = \{x \in \Omega :\ p(x) \le n\}.$$

Thus $p_+(\Omega^{(n)}) \le n < \infty$ on $\Omega^{(n)}$. Using (4.1) for $L^{p(x)}(\Omega^{(n)})$, there exists a $u \in L^{p(x)}(\Omega^{(n)})$ such that $\int_{\Omega^{(n)}} |u(x)|^{p(x)} d\mu = 1$ and $\|u\|^o_{p(x)|\Omega^{(n)}} \ge d_{(p_-,\, p_+)} - \varepsilon$. Let

$$\widetilde{u}(x) = \begin{cases} u(x), & \text{if} \quad x \in \Omega^{(n)}, \\ 0, & \text{if} \quad x \in \Omega\backslash\Omega^{(n)}. \end{cases}$$

Then $\widetilde{u}(x) \in L^{p(x)}(\Omega)$ and $\int_\Omega |\widetilde{u}(x)|^{p(x)} d\mu = 1$. Thus $\|\widetilde{u}(x)\|_{p(x)} = 1$. Clearly,

$$\|u\|^o_{p(x)|\Omega^{(n)}} = \|\widetilde{u}(x)\|_{p(x)}.$$

It means

$$\sup_{u\in L^{p(x)}(\Omega)\backslash\{0\}} \frac{\|u\|^o_{p(x)}}{\|u\|_{p(x)}} \ge d_{(p_-,\, n)} - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\sup_{u\in L^{p(x)}(\Omega)\backslash\{0\}} \frac{\|u\|^o_{p(x)}}{\|u\|_{p(x)}} \ge d_{(p_-,\, n)}, \quad \forall\ n.$$

Thus

$$\sup_{u\in L^{p(x)}(\Omega)\backslash\{0\}} \frac{\|u\|^o_{p(x)}}{\|u\|_{p(x)}} \ge \lim_{n\to\infty} d_{(p_-,\, n)} = d_{(p_-,\, \infty)}.$$

The proof is complete.

Lemma 4.4. (4.1) holds when $\Omega_\infty \ne \emptyset$.

Proof. We assume $\Omega\backslash\Omega_\infty \ne \emptyset$ since when $\Omega = \Omega_\infty$ (4.1) is obviously true (see Remark 1.1). Given any $\varepsilon > 0$, let $A_- \subset \Omega\backslash\Omega_\infty$ such that $\mu(A_-) > 0$ and

$$\left|\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}} k^{p(x)} - \frac{(p_- - 1)^{p_- - 1}}{p_-^{p_-}} k^{p_-}\right| \le \frac{\varepsilon}{2}, \quad \forall\ x \in A_-, \quad \forall\ k \in \left[\frac{1}{2},\ 1\right].$$

Let

$$u(x)=\begin{cases}1, & \text{if} \quad x\in\Omega_\infty\,,\\ a, & \text{if} \quad x\in A_-\,,\\ 0, & \text{if} \quad x\in\Omega\backslash(A_-\cup\Omega_\infty),\end{cases}$$

where the positive constant a satisfies $\int_{A_-} a^{p(x)}d\mu=1$. Then we have $\|u\|_{p(x)}=1$ and

$$\begin{aligned}\|u\|^o_{p(x)}&=\inf_{k>0}\frac{1}{k}\Big(\int_{\Omega\backslash\Omega_\infty}\Phi(x,\ |ku(x)|)d\mu+\int_{\Omega_\infty}\Phi(x,\ |ku(x)|)d\mu+1\Big)\\ &=\inf_{\frac12\le k\le 1}\frac{1}{k}\Big(\int_{A_-}\frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}}\,k^{p(x)}|u(x)|^{p(x)}d\mu+1\Big)\\ &\ge\inf_{\frac12\le k\le 1}\frac{1}{k}\Big(\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,k^{p_-}+1-\frac{\varepsilon}{2}\Big)\\ &\ge\inf_{\frac12\le k\le 1}\frac{1}{k}\Big(\frac{(p_--1)^{p_--1}}{p_-^{p_-}}\,k^{p_-}+1\Big)-\varepsilon\\ &\ge\frac{(p_--1)^{p_--1}}{p_-^{p_-}}+1-\varepsilon=d_{(p_-,\ \infty)}-\varepsilon.\end{aligned}\tag{4.9}$$

Thus (4.1) holds.

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