Linear Algebra

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This is a tutorial demo training to all new Teaching Assistants. It contains ten typical problems in Linear Algebra. The new TA would be randomly assigned a problem, and present the problem on blackboard to other new TAs. Pros and Cons of each presentation would be highlighted and discussed after the presentation. Typical teaching techniques/tricks/mistakes would also be emphasised.

Problem 1. Solve the following system of linear equations:

$$\begin{cases} 2x_1 & +7x_2 & +3x_3 & +x_4 & = & 6 \\ 3x_1 & +5x_2 & +2x_3 & +2x_4 & = & 4 \\ 9x_1 & +4x_2 & +x_3 & +7x_4 & = & 2 \end{cases}$$

Keywords. Elementary row operations, rank, solution structures of linear system of equations.

Suggested Solution. Let A be the coefficient matrix. We do elementary row operations to the augmented matrix $\bar{A}$:

$$\bar{A} = \begin{pmatrix} 2 & 7 & 3 & 1 & 6 \\ 3 & 5 & 2 & 2 & 4 \\ 9 & 4 & 1 & 7 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & 1 & -1 & 2 \\ 0 & 11 & 5 & -1 & 10 \\ 0 & -11 & -5 & 1 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 1 & -2 \\ 0 & 1 & \frac{5}{11} & -\frac{1}{11} & \frac{10}{11} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -\frac{1}{11} & \frac{9}{11} & -\frac{2}{11} \\ 0 & 1 & \frac{5}{11} & -\frac{1}{11} & \frac{10}{11} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\operatorname{rank}(A) = \operatorname{rank}(\bar{A}) = 2 < n = 4$, the system has infinite many solutions. Let $x_3 = x_4 = 0$, we find $x_1 = -\frac{2}{11}$, $x_2 = \frac{10}{11}$, and a special solution

$$\eta = \begin{pmatrix} -\frac{2}{11} \\ \frac{10}{11} \\ 0 \\ 0 \end{pmatrix}.$$

Let $x_3 = 1$, $x_4 = 0$, we obtain $x_1 = \frac{1}{11}$, $x_2 = -\frac{5}{11}$. Let $x_3 = 0$, $x_4 = 1$, we obtain $x_1 = -\frac{9}{11}$, $x_2 = \frac{1}{11}$. Let

$$\xi_1 = \begin{pmatrix} \frac{1}{11} \\ -\frac{5}{11} \\ 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -\frac{9}{11} \\ \frac{1}{11} \\ 0 \\ 1 \end{pmatrix}.$$

The general solutions of the system are $x = \eta + k_1\xi_1 + k_2\xi_2$ for arbitrary numbers k_1 and k_2. ∎

Problem 2. For various cases of the two numbers a and b, find out the solution(s) of the following system of linear equations:

$$\begin{cases} x_1 & +x_2 & -x_3 & = & 1 \\ 2x_1 & +(a+2)x_2 & -(b+2)x_3 & = & 3 \\ & -3ax_2 & +(a+2b)x_3 & = & -3 \end{cases}$$

Keywords. Determinant, Cramer's rule, elementary row operations.
Suggested Solution. The number of equations and the number of unknowns are the same. Let A be the coefficient matrix of the system. Then

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & a+2 & -(b+2) \\ 0 & -3a & a+2b \end{vmatrix} = a(a-b).$$

We discuss the following three cases.

1). $|A| \neq 0$, i.e., $a \neq 0$ and $a \neq b$. In this case, the system has a unique solution. We can use Cramer's rule to find it out.

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -(b+2) \\ 0 & -3 & a+2b \end{vmatrix}}{a(a-b)} = \frac{1}{a}, \quad x_3 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & a+2 & 3 \\ 0 & -3a & -3 \end{vmatrix}}{a(a-b)} = 0, \quad x_1 = 1 - x_2 + x_3 = 1 - \frac{1}{a}.$$

2). $a = 0$. In this case we do elementary row operations to the augmented matrix:

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 2 & -(b+2) & 3 \\ 0 & 0 & 2b & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & -b & 1 \\ 0 & 0 & 2b & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & -b & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So the system has no solution.

3). $a = b \neq 0$. We do elementary row operations to the augmented matrix:

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & a+2 & -(a+2) & 3 \\ 0 & -3a & 3a & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & a & -a & 1 \\ 0 & -3a & 3a & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & \frac{1}{a} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{a} \\ \frac{1}{a}+k \\ k \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{a} \\ \frac{1}{a} \\ 0 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

where k is an arbitrary number.

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Problem 3. Let

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 5 \end{pmatrix}, \ \alpha_3 = \begin{pmatrix} 1 \\ -1 \\ a+2 \\ 1 \end{pmatrix}, \ \alpha_4 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ a+8 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 \\ 1 \\ b+3 \\ 5 \end{pmatrix}.$$

1). For which values of a and b will β **not** be represented as linear combination of α_1, α_2, α_3, α_4.

2). For which values of a and b will β be uniquely represented as linear combination of α_1, α_2, α_3, α_4.

Keywords. Linear combination of vectors, solution structure of linear system of equations, augmented matrix, elementary row operations, rank.

Suggested Solution. By the definition, whether β can be represented as linear combination of α_1, α_2, α_3, α_4 or not is equivalent to whether the system

$$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 = \beta$$

has solution or not. Let us solve the system:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ \quad\ \ x_2 - x_3 + 2x_4 = 1 \\ 2x_1 + 3x_2 + (a+2)x_3 + 4x_4 = b+3 \\ 3x_1 + 5x_2 + x_3 + (a+8)x_4 = 5 \end{cases}$$

Let the coefficient matrix be A. We do elementary row operations on the augmented matrix $\bar{A}$:

$$\begin{aligned}\bar{A} &= \begin{pmatrix}1&1&1&1&1\\0&1&-1&2&1\\2&3&a+2&4&b+3\\3&5&1&a+8&5\end{pmatrix}\to\begin{pmatrix}1&1&1&1&1\\0&1&-1&2&1\\0&1&a&2&b+1\\0&2&-2&a+5&2\end{pmatrix}\\ &\to \begin{pmatrix}1&1&1&1&1\\0&1&-1&2&1\\0&0&a+1&0&b\\0&0&0&a+1&0\end{pmatrix}.\end{aligned}$$

1). When $a=-1$, $b\neq 0$, $\text{rank}(A)=2<3=\text{rank}(\bar{A})$, the system has no solution. In this case, β cannot be represented as linear combination of α_1, α_2, α_3, α_4.

2). When $a\neq -1$, for arbitrary b we have $\text{rank}(A)=\text{rank}(\bar{A})=4$, the system has a unique solution:

$$\begin{cases}x_1 &= -\frac{2b}{a+1}\\ x_2 &= 1+\frac{b}{a+1}\\ x_3 &= \frac{b}{a+1}\\ x_4 &= 0\end{cases}$$

In this case,

$$\beta=-\frac{2b}{a+1}\alpha_1+(1+\frac{b}{a+1})\alpha_2+\frac{b}{a+1}\alpha_3.$$

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Problem 4. Let $A=\begin{pmatrix}1-\sin(\theta)\cos(\theta) & \cos^2(\theta)\\ -\sin^2(\theta) & 1+\sin(\theta)\cos(\theta)\end{pmatrix}$.

1). Find eigenvalues of A.

2). Find the corresponding eigenvectors.

3). Find an invertible 2×2 matrix P such that $\tilde{A}=P^{-1}AP$ is in Jordan normal form.

Keywords. Eigenvalue, eigenvector, Jordan normal form.
Suggested Solution.

1).

$$|A-\lambda I|=\begin{vmatrix}1-\sin(\theta)\cos(\theta)-\lambda & \cos^2(\theta)\\ -\sin^2(\theta) & 1+\sin(\theta)\cos(\theta)-\lambda\end{vmatrix}=(\lambda-1)^2.$$

The eigenvalue of A is 1, with mutiplicity 2.

2). Let us solve the equation $(A-\lambda I)x=0$ for $\lambda=1$.

$$\begin{pmatrix}-\sin(\theta)\cos(\theta) & \cos^2(\theta)\\ -\sin^2(\theta) & \sin(\theta)\cos(\theta)\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Longrightarrow p_1=\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}.$$

So the eigenvector corresponding the the eigenvalue 1 is p_1 (or nonzero scalar multiplication of p_1).

3). We would like to find a vector p_2 satisfying $Ap_2 = p_1 + \lambda p_2$, i.e., $(A - \lambda I)p_2 = p_1$, where $\lambda = 1$. Solving this equation, we obtain $p_2 = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$.

Let $P = (p_1, p_2) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. Then

$$AP = (Ap_1, Ap_2) = (\lambda p_1, p_1 + \lambda p_2) = (p_1, p_2)\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = P\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

So $\tilde{A} = P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

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Problem 5. Let $A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$.

1). Diagonalize A.

2). Let t be a formal variable, compute $(tA)^2$, $(tA)^3$.

3). Define $\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n$ as the formal sum (regardless the convergence). Compute $\exp(tA)$.

Keywords. Matrix diagonalization, eigenvalue, eigenvector, exponential of matrix, application of linear algebra.
Suggested Solution.

1). Let us compute the eigenvalues and corresponding eigenvectors.

$$\begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 14 = (\lambda+2)(\lambda-7), \Longrightarrow \lambda_1 = -2,\ \lambda_2 = 7.$$

For $\lambda_1 = -2$, we have

$$\begin{pmatrix} 5 & 4 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}.$$

For $\lambda_1 = 7$, we have

$$\begin{pmatrix} -4 & 4 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $P = \begin{pmatrix} 4 & 1 \\ -5 & 1 \end{pmatrix}$, we have

$$P^{-1}AP = \begin{pmatrix} -2 & 0 \\ 0 & 7 \end{pmatrix}.$$

Here $P^{-1} = \begin{pmatrix} \frac{1}{9} & -\frac{1}{9} \\ \frac{5}{9} & \frac{4}{9} \end{pmatrix}$.

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2).

$$tA = P\begin{pmatrix} -2t & 0 \\ 0 & 7t \end{pmatrix} P^{-1} \quad \Longrightarrow$$

$$(tA)^2 = P\begin{pmatrix} -2t & 0 \\ 0 & 7t \end{pmatrix} P^{-1} P\begin{pmatrix} -2t & 0 \\ 0 & 7t \end{pmatrix} P^{-1} = P\begin{pmatrix} (-2t)^2 & 0 \\ 0 & (7t)^2 \end{pmatrix} P^{-1} = \begin{pmatrix} 29t^2 & 20t^2 \\ 25t^2 & 24t^2 \end{pmatrix}.$$

Similarly, $(tA)^3 = P\begin{pmatrix} (-2t)^3 & 0 \\ 0 & (7t)^3 \end{pmatrix} P^{-1} = \begin{pmatrix} 187t^3 & 156t^3 \\ 195t^3 & 148t^3 \end{pmatrix}$.

3). We find that

$$(tA)^n = P\begin{pmatrix} (-2t)^n & 0 \\ 0 & (7t)^n \end{pmatrix} P^{-1}.$$

So we obtain

$$\begin{aligned} \exp(tA) &= \sum_{n=0}^{\infty} \frac{1}{n!} P\begin{pmatrix} (-2t)^n & 0 \\ 0 & (7t)^n \end{pmatrix} P^{-1} = P\begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}(-2t)^n & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!}(7t)^n \end{pmatrix} P^{-1} \\ &= P\begin{pmatrix} \exp(-2t) & 0 \\ 0 & \exp(7t) \end{pmatrix} P^{-1} \\ &= \frac{1}{9}\begin{pmatrix} 4\exp(-2t)+5\exp(7t) & -4\exp(-2t)+4\exp(7t) \\ -5\exp(-2t)+5\exp(7t) & 5\exp(-2t)+4\exp(7t) \end{pmatrix}. \end{aligned}$$

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Remark. When you learn some knowledge on linear differential equations, you will see above are typical processes on solving the equation $x'(t) = Ax(t)$, where $x(t)$ is a unknown vector function of t. The general solution is $x(t) = \exp(tA)x_0$, where x_0 is given as the initial vector. This is an application of using linear algebra to solving ODE, and you already see the power of matrix diagonalization.

Problem 6. Let $u = (6, a+1, 3)^{\mathrm{T}}$, $v = (a, 2, -2)^{\mathrm{T}}$, $w = (a, 1, 0)^{\mathrm{T}}$.

1). For which value of a will u and v be linear dependent or linear independent? When they are linear dependent, write down their linear relation.

2). For which value of a will u, v and w be linear dependent or linear independent? When they are linear dependent, write down their linear relation.

Keywords. Linear dependence/independence, elementary row operations, determinant.
Suggested Solution.

1). Let $x_1 u + x_2 v = 0$, we obtain a system of linear equations:

$$\begin{cases} 6x_1 & +ax_2 & = & 0 \\ (a+1)x_1 & +2x_2 & = & 0 \\ 3x_1 & -2x_2 & = & 0 \end{cases}$$

6

We do elementary row operations to the augmented matrix:

$$\begin{pmatrix} 6 & a & 0 \\ a+1 & 2 & 0 \\ 3 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 6 & a & 0 \\ 0 & -\frac{a^2+a-12}{6} & 0 \\ 0 & -\frac{a}{2}-2 & 0 \end{pmatrix}.$$

When

$$\begin{cases} -\frac{a^2+a-12}{6} & = & 0 \\ -\frac{a}{2}-2 & = & 0 \end{cases}$$

i.e., $a=-4$, the system has nonzero solutions hence u and v are linear dependent. And we can take $x_1=2$, $x_2=3$, so $2u+3v=0$. When $a\neq -4$, the system only has zero solution hence u and v are linear independent.

2). Let us compute the determinant of matrix generated by u, v, w:

$$\begin{vmatrix} 6 & a & a \\ a+1 & 2 & 1 \\ 3 & -2 & 0 \end{vmatrix} = -2a^2-5a+12 = -(a+4)(2a-3).$$

When $a=-4$ or $a=\frac{3}{2}$, u, v, w are linear dependent.

i). The case $a=\frac{3}{2}$. Let us write $x_1u+x_2v+x_3w=0$. We do elementary row operations for the augmented matrix:

$$\begin{pmatrix} 6 & \frac{3}{2} & \frac{3}{2} & 0 \\ \frac{5}{2} & 2 & 1 & 0 \\ 3 & -2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & 1 & \frac{3}{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We may take $x_3=11$, $x_2=-3$, $x_1=-6$, i.e., $-6u-3v+11w=0$.

ii). The case $a=-4$. We still use the relation as step 1): $2u+3v+0w=0$.

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Problem 7. Suppose that A is a $n\times n$ matrix with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$, and the corresponding eigenvectors $\xi_1,\xi_2,\ldots,\xi_n$.

1). What are the eigenvalues and the corresponding eigenvectors of kA (k is a nonzero constant)?

2). What are the eigenvalues and the corresponding eigenvectors of A^m (m is a positive integer)?

3). Assume that A is invertible. What are the eigenvalues and the corresponding eigenvectors of A^{-1}?

4). Assume that A is invertible. Let $\mathrm{adj}(A)$ be the adjugate matrix of A. So we have the Laplace's formula $A\,\mathrm{adj}(A)=\mathrm{adj}(A)\,A=|A|I_n$. What are the eigenvalues and the corresponding eigenvectors of $\mathrm{adj}(A)$?

5). Let P be a $n \times n$ invertible matrix. What are the eigenvalues and the corresponding eigenvectors of $P^{-1}AP$?

6). Let $f(x) = x^m + c_1x^{m-1} + \cdots + c_{m-1}x + c_m$ (m is a positive integer). What are the eigenvalues and the corresponding eigenvectors of $f(A)$?

7). What are the eigenvalues of A^{T}?

Keywords. Eigenvalue, eigenvector, adjugate matrix, matrix polynomial, determinant.
Suggested Solution. By definition we have $A\xi_i = \lambda_i\xi_i$.

1). Since $kA\xi_i = k\lambda_i\xi_i$, kA has eigenvalues $k\lambda_1$, $k\lambda_2$, $\ldots$, $k\lambda_n$, and the corresponding eigenvectors $\xi_1, \xi_2, \ldots, \xi_n$.

2). Since $A^m\xi_i = A^{m-1}A\xi_i = \lambda_i A^{m-1}\xi_i = \cdots = \lambda_i{}^m\xi_i$, A^m has eigenvalues $\lambda_1{}^m$, $\lambda_2{}^m$, $\ldots$, $\lambda_n{}^m$, and the corresponding eigenvectors $\xi_1, \xi_2, \ldots, \xi_n$.

3). Since A is invertible, $|A| = \lambda_1\lambda_2\cdots\lambda_n \neq 0$. Each λ_i is nonzero. Left multiplying $\lambda_i{}^{-1}A^{-1}$ to the equation $A\xi_i = \lambda_i\xi_i$, we obtain $A^{-1}\xi_i = \lambda_i{}^{-1}\xi_i$. So A^{-1} has eigenvalues $\lambda_1{}^{-1}$, $\lambda_2{}^{-1}$, $\ldots$, $\lambda_n{}^{-1}$, and the corresponding eigenvectors $\xi_1, \xi_2, \ldots, \xi_n$. Therefore, when A is invertible, we can only assume that m is a nonzero integer in step 2) and still obtain the same results.

4). By the Laplace formula, we obtain $\mathrm{adj}A = |A|A^{-1}$. By step 1) and 3), A has eigenvalues $\frac{|A|}{\lambda_1}$, $\frac{|A|}{\lambda_2}$, $\ldots$, $\frac{|A|}{\lambda_n}$ and the corresponding eigenvectors $\xi_1, \xi_2, \ldots, \xi_n$.

5). We have $APP^{-1}\xi_i = \lambda_i\xi_i$. Left multiplying P^{-1}, we get $(P^{-1}AP)(P^{-1}\xi_i) = \lambda_i(P^{-1}\xi_i)$. So $P^{-1}AP$ has eigenvalues λ_1, λ_2, $\ldots$, λ_n, and the corresponding eigenvectors $P^{-1}\xi_1$, $P^{-1}\xi_2$, $\ldots$, $P^{-1}\xi_n$.

6). By step 1) and 2), we obtain that $f(A)$ has eigenvalues $f(\lambda_1)$, $f(\lambda_2)$, $\ldots$, $f(\lambda_n)$, and the corresponding eigenvectors $\xi_1, \xi_2, \ldots, \xi_n$.

7). Since $|(A^{\mathrm{T}} - \lambda I)| = |(A - \lambda I)^{\mathrm{T}}| = |(A - \lambda I)|$, so A^{T} and A have the same eigenvalues.

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Problem 8. Let A be the real symmetric matrix $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. Find an orthogonal matrix T to diagonalize A.
Keywords. Real symmetric matrix, orthogonal matrix, inner product, Gram-Schmidt process.
Suggested Solution. We first find the eigenvalues and the corresponding eigenvectors of A.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 8-\lambda & 2 & 2 \\ 8-\lambda & 4-\lambda & 2 \\ 8-\lambda & 2 & 4-\lambda \end{vmatrix} = (8-\lambda)\begin{vmatrix} 1 & 2 & 2 \\ 1 & 4-\lambda & 2 \\ 1 & 2 & 4-\lambda \end{vmatrix} \\ &= (8-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (8-\lambda)(2-\lambda)^2. \end{aligned}$$

- For $\lambda_1 = 8$, we have
$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
We obtain an eigenvector $\xi_1 = (1,1,1)^{\mathrm{T}}$.

- For $\lambda_2 = \lambda_3 = 2$(multiplicity 2), we have
$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
We obtain two linear independent eigenvectors $\xi_2 = (-1,1,0)^{\mathrm{T}}$, $\xi_3 = (-1,0,1)^{\mathrm{T}}$.

ξ_1 is already orthogonal to ξ_2 and ξ_3 (this is not a coincidence, why?). Next we apply Gram-Schmidt process for ξ_2 and ξ_3. Take
$$\eta_2 = \xi_2,$$
$$\eta_3 = \xi_3 - \frac{(\xi_3, \eta_2)}{(\eta_2, \eta_2)} \eta_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$
We normalize ξ_1, η_2, η_3 and obtain
$$\alpha_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}.$$
Let
$$T = (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$
Then
$$T^{-1} A T = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

∎

Remark.

1). We have many ways to choose ξ_2 and ξ_3 so the matrix T is not unique.

2). For a real symmetry matrix A, if $A\xi = \lambda_1 \xi$, $A\eta = \lambda_2 \eta$, and $\lambda_1 \neq \lambda_2$, then ξ and η are orthogonal. The proof is as following: $\lambda_1(\xi, \eta) = (A\xi, \eta) = (\xi, A\eta) = \lambda_2(\xi, \eta)$, and $\lambda_1 \neq \lambda_2$, hence $(\xi, \eta) = 0$.

Problem 9. Let $C[0,1]$ be the set of all real valued continuous functions on the interval $[0,1]$.

1). For any $f,\ g, \in C[0,1]$ and a scalar $a \in \mathbb{R}$, define

$$\begin{aligned}(f+g)(t) &= f(t)+g(t),\\ (af)(t) &= af(t).\end{aligned}$$

Check that the $C[0,1]$ with above addition and scalar multiplication is a vector space over $\mathbb{R}$. This is an example of infinite-dimensional vector space.

2). For any $f,\ g \in C[0,1]$, define $\langle f,g\rangle = \int_0^1 f(t)g(t)dt$. Check that $\langle\cdot,\cdot\rangle$ is an inner product on the vector space $C[0,1]$.

3). Let V be the subspaces of functions generated by the two functions $f(t)=t$, $g(t)=t^2$. Find an orthonormal basis for V.

Keywords. Example of vector space, inner product, orthonormal basis, Gram-Schmidt process, application of linear algebra.
Suggested Solution.

1). For any $f,\ g,\ h \in C[0,1]$, and $a,b \in \mathbb{R}$, we check the following axioms.
   - Associativity of addition: $f+(g+h) = (f+g)+h$.
   - Commutativity of addition: $f+g = g+f$
   - Identity element of addition: there exists an element $\mathbf{0} \in C[0,1]$, called the zero vector, such that $f+\mathbf{0} = f$. Here $\mathbf{0}$ is the function which maps every element in $[0,1]$ to the value 0.
   - Inverse elements of addition: for every $f \in C[0,1]$, there exists an element $-f \in C[0,1]$, called the additive inverse of f, such that $f+(-f) = \mathbf{0}$.
   - Distributivity of scalar multiplication with respect to vector addition: $a(f+g) = af+ag$.
   - Distributivity of scalar multiplication with respect to field addition:$(a+b)f = af+bf$.
   - Compatibility of scalar multiplication with field multiplication: $a(bf) = (ab)f$.
   - Identity element of scalar multiplication: $1f = f$, where 1 denotes the multiplicative identity in $\mathbb{R}$.

2). We can check easily that:
   - $\langle f,g\rangle = \int_0^1 f(t)g(t)dt = \langle g,f\rangle$.
   - $\langle af,g\rangle = \int_0^1 af(t)g(t)dt = a\langle f,g\rangle$.
   - $\langle f+g,h\rangle = \int_0^1 (f(t)+g(t))h(t)dt = \langle f,h\rangle + \langle g,h\rangle$.
   - $\langle f,f\rangle = \int_0^1 f(t)f(t)dt \geq 0$, with equality if and only if $f=0$.

3). We apply Gram-Schmidt process to $f(t) = t$ and $g(t) = t^2$. Let $h = g - \frac{\langle g,f\rangle}{\langle f,f\rangle} f$.

$$\|f\|^2 = \langle f, f\rangle = \int_0^1 t^2 dt = \frac{1}{3}, \quad h(t) = t^2 - 3(\int_0^1 t^3 dt)t = t^2 - \frac{3}{4}t.$$

Let us normalize f and h: $\|h\|^2 = \int_0^1 (t^2 - \frac{3}{4}t)^2 dt = \frac{1}{80}$. So $\frac{f}{\|f\|} = \frac{\sqrt{3}t}{3}$ and $\frac{h}{\|h\|} = \sqrt{80}(t^2 - \frac{3}{4}t)$ will be an orthonormal basis for V.

∎

Remark. We can play exactly the same game for the space $C[-\pi,\pi]$ with inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$. Let n be integer. Then

$$g_n = \sin(nt),\ n > 0 \quad \text{and} \quad h_n = \cos(nt),\ n \geq 0$$

will form an orthonormal basis. For example, we can express t in term of above basis. This is the linear algebra theory parts of the Fourier analysis, and it is easy. The difficulty parts are how to handle the infinite sum, i.e., the convergent problem and analysis.

Problem 10.

1). Let $V = \mathbb{R}^3$ and take the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Let $P : V \to V$ such that $P(\mathbf{e}_1) = \mathbf{e}_1$, $P(\mathbf{e}_2) = \mathbf{e}_2$, $P(\mathbf{e}_3) = 0$. Write down the matrix A which represents the operator P in the standard basis. Check that $P^2 = P$ (here P^2 means $P \circ P : V \to V$) and $A^2 = A$. What are the eigenvalues of P?

2). Let V be a vector space and $P : V \to V$ be a linear transform such that $P^2 = P$.

i). What are the possible eigenvalues of P?

ii). Let $U = \ker(P)$, $W = \mathrm{im}(P)$. Let $Q = I - P : V \to V$.

- Check that P is the zero operator on U and the identity operator on W.
- Check that $Q^2 = Q$.
- Check that Q is the zero operator on W and the identity operator on U.
- Check that for any $x \in V$, we can write $x = u + w$ for some $u \in U$ and $w \in W$.
- Check that the above decomposition is unique. Therefore $V = U \oplus W$.

Keywords. Linear transform, projection operator, kernel, image, decomposition of space, invariant subspace

Suggested Solution.

1). We would like to find a matrix A such that

$$P(x) = Ax, \forall x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Since

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

We have

$$P(x) = P(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (P\mathbf{e_1}, P\mathbf{e_2}, P\mathbf{e_3}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

So we take

$$A = (P\mathbf{e_1}, P\mathbf{e_2}, P\mathbf{e_3}) = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $P^2 = P$ and $A^2 = A$. It is obvious that P has eigenvalue 1 with the corresponding eigenvectors $\mathbf{e_1}, \mathbf{e_2}$ and eigenvalue 0 with the corresponding eigenvector $\mathbf{e_3}$.

2). i). Suppose that $P(x) = \lambda x$ for some nonzero vector x. Then $\lambda x = P(x) = P^2(x) = P(\lambda x) = \lambda^2 x$, so $(\lambda^2 - \lambda)x = 0$. Since x is nonzero, the possible eigenvalues are $\lambda = 1$ or $\lambda = 0$.

ii). We already know that both the kernel U and the image W are invariant subspaces of V.

* By definition of kernel, for any $u \in U$, $P(u) = 0$. Hence the restriction of P to the subspace U is the zero operator. By the definition of image, for any $w \in W$, we can write $w = P(v)$ for some vector $v \in V$. So

$$P(w) = P(P(v)) = P^2(v) = P(v) = w,$$

and the restriction of P to the subspace W is the identity operator.

* $Q^2 = (I - P)^2 = I - 2P + P^2 = I - P = Q$.
* For any $w \in W$ we have $Q(w) = w - P(w) = w - w = 0$. So the restriction of Q on W is the zero operator. For any $u \in U$ we have $Q(u) = (I - P)(u) = u$. So the restriction of Q on U is the identity operator.
* For any $x \in V$, we can write $x = (x - P(x)) + P(x)$ where $u = x - P(x) = Q(x) \in U$ and $w = P(x) \in W$.
* If $x = \tilde{u} + \tilde{w}$ for some $\tilde{u} \in U$ and $\tilde{w} \in W$, we have $u = Q(x) = Q(\tilde{u} + \tilde{w}) = \tilde{u}$ and $w = P(x) = P(\tilde{u} + \tilde{w}) = \tilde{w}$. So the decomposition is unique.

$\blacksquare$

Remark Geometrically, we have the projection operator which is a linear operator. You can think above P as the projection of V onto W along the "direction" U; and Q as the projection of V onto U along the "direction" W. Projection twice is the same as projection once, so we have $P^2 = P$ and $Q^2 = Q$. If we have a nontrivial projection, which means none of P nor Q is the identity operator on V, then we can decompose V into direct sum of smaller spaces as above. Algebraically, if a linear operator (or a matrix) $P : V \to V$ satisfies $P^2 = P$, we just call P a projection. Such a matrix is called an idempotent matrix.

Calculus I

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This is a tutorial demo training to all new Teaching Assistants. It contains ten typical problems in one variable Calculus. The new TA would be randomly assigned a problem, and present the problem on blackboard to other new TAs. Pros and Cons of each presentation would be highlighted and discussed after the presentation. Typical teaching techniques/tricks/mistakes would also be emphasised.

Problem 1. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0; \\ 0 & x = 0. \end{cases}$$

1). Show that $f(x)$ is continuous at 0.

2). Show that $f(x)$ is differentiable, and compute $f'(x)$.

3). Show $f'(x)$ is continuous at 0.

Keywords. Continuity, limit, differentiability, example of infinitely differentiable function.
Suggested Solution.

1). By taking $t = \frac{1}{x^2}$, we have

$$\lim_{x\to 0} f(x) = \lim_{t\to+\infty} \frac{1}{e^t} = 0 = f(0).$$

So $f(x)$ is continuous at $x = 0$.

2). When $x \neq 0$, $f(x)$ is the composition of some basic function and hence differentiable. We need to check that $\lim_{\Delta x\to 0} \frac{f(0+\Delta x)-f(0)}{\Delta x - 0}$ exists.

$$\begin{aligned}
\lim_{\Delta x\to 0} \frac{f(0+\Delta x)-f(0)}{\Delta x-0} &= \lim_{\Delta x\to 0} \frac{e^{-\frac{1}{\Delta x^2}}}{\Delta x} \\
&= \lim_{t\to\infty} \frac{t}{e^{t^2}} \quad \text{taking } t = \frac{1}{\Delta x} \\
&= \lim_{t\to\infty} \frac{1}{2te^{t^2}} \quad \text{L'Hôpital's rule} \\
&= 0,
\end{aligned}$$

so $f'(0) = 0$. Hence

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-\frac{1}{x^2}} & x \neq 0; \\ 0 & x = 0. \end{cases}$$

3).

$$\begin{aligned}
\lim_{x\to 0} f'(x) &= \lim_{x\to 0} \frac{2}{x^3} e^{-\frac{1}{x^2}} \\
&= \lim_{t\to\infty} \frac{2t^3}{e^{t^2}} \quad \text{taking } t = \frac{1}{x} \\
&= \lim_{t\to\infty} \frac{6t^2}{2te^{t^2}} = \lim_{t\to\infty} \frac{12t}{(2te^{t^2})'} = \lim_{t\to\infty} \frac{12}{(2te^{t^2})''} \quad \text{L'Hôpital's rule three times} \\
&= 0.
\end{aligned}$$

So $\lim_{x\to 0} f'(x) = 0 = f'(0)$, and $f'(x)$ is continuous.

∎

Remark. By the mathematical induction, it can be shown that $f(x)$ is infinitely differentiable at $x = 0$, and $f^{(n)}(0) = 0$ for any positive integer n.

Problem 2. Let α and β be two positive numbers, and

$$f(x) = \begin{cases} x^\alpha \sin(\frac{1}{x^\beta}) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

1). Show that $f(x)$ is continuous at 0.

2). For what values of α and β is $f(x)$ differentiable?

3). For what values of α and β is $f'(x)$ continuous?

Keywords. Continuity, differentiability, example of differentiable function but its differential is not continuous.
Suggested Solution.

1). Since $\sin(\frac{1}{x^\beta})$ is bounded, it is clear that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^\alpha \sin(\frac{1}{x^\beta}) = 0 = f(0).$$

2). When $x \neq 0$, we have

$$\begin{aligned} f'(x) &= \alpha x^{\alpha-1}\sin(\frac{1}{x^\beta}) + x^\alpha \cos(\frac{1}{x^\beta})(\frac{-\beta}{x^{\beta+1}}) \\ &= \alpha x^{\alpha-1}\sin(\frac{1}{x^\beta}) - \beta x^{\alpha-\beta-1}\cos(\frac{1}{x^\beta}). \end{aligned}$$

We need to compute $\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x - 0}$.

$$\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x - 0} = \lim_{\Delta x\to 0} \Delta x^{\alpha-1}\sin(\frac{1}{\Delta x^\beta}).$$

When $\alpha > 1$, the above limit exists and $f'(0) = 0$. When $0 < \alpha \leq 1$, the limit does not exist. Therefore, for $\alpha > 1$ and arbitrary positive β, $f(x)$ is differentiable and

$$f'(x) = \begin{cases} \alpha x^{\alpha-1}\sin(\frac{1}{x^\beta}) - \beta x^{\alpha-\beta-1}\cos(\frac{1}{x^\beta}) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

When $0 < \alpha \leq 1$, $f(x)$ is not differentiable.

3). We need to compute $\lim_{x\to 0} f'(x)$. When $\alpha > 1$ and $\alpha - \beta - 1 > 0$, the limit exists and equals 0, then $f'(x)$ is continuous. When $\alpha > 1$, and $\alpha - \beta - 1 \leq 0$, $f'(x)$ exists but is not continuous.

∎

Remark. We can further ask similar questions. For what values of positive α and β does $f''(x)$ exist? For what values of positive α and β is $f''(x)$ continuous?

Problem 3. Find the limit

$$\lim_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + \cdots + a_n^{\frac{1}{x}}}{n}\right)^{nx}$$

where $a_1,\ a_2,\ \cdots,\ a_n$ are positive.
Keywords. Limit, L'Hôpital's rule.

Suggested Solution.

$$\begin{aligned}
&\lim_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}}}{n}\right)^{nx}\\
=\;&\lim_{x\to\infty}\exp\left(nx\ln\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}}}{n}\right)\\
=\;&\exp\left(n\lim_{x\to\infty}\frac{\ln(a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}})-\ln n}{\frac{1}{x}}\right)\quad\text{continuity of limit}\\
=\;&\exp\left(n\lim_{x\to\infty}\frac{\frac{a_1^{\frac{1}{x}}\ln a_1+a_2^{\frac{1}{x}}\ln a_2+\cdots+a_n^{\frac{1}{x}}\ln a_n}{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}}}(-\frac{1}{x^2})}{-\frac{1}{x^2}}\right)\quad\text{L'Hôpital's rule}
\end{aligned}$$

Since

$$\lim_{x\to\infty}a^{\frac{1}{x}}=1\quad\text{for } a>0,$$

we have

$$\lim_{x\to\infty}\frac{a_1^{\frac{1}{x}}\ln a_1+a_2^{\frac{1}{x}}\ln a_2+\cdots+a_n^{\frac{1}{x}}\ln a_n}{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}}}=\frac{\ln(a_1a_2\cdots a_n)}{n}.$$

So the original limit equals

$$\exp\left(n\frac{\ln(a_1a_2\cdots a_n)}{n}\right)=a_1a_2\cdots a_n.$$

∎

Problem 4. Find the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.
1). $y=\tan(x+y)$. 2). $y=\left(\frac{x}{1+x}\right)^x$.
3). $\begin{cases}x&=&f'(t)\\y&=&tf'(t)-f(t)\end{cases}$ where we assume that $f''(t)$ exists and does not equal zero.
Keywords. Implicit function, chain rule, parameter function, logarithmic differentiation.
Suggested Solution.

1). This is an implicit function of y. Taking derivative with respect to x on both sides, we have

$$y'=\sec^2(x+y)(1+y').$$

So

$$y'=\frac{-\sec^2(x+y)}{\sec^2(x+y)-1}=\frac{-\sec^2(x+y)}{\tan^2(x+y)}=-\frac{1+y^2}{y^2}=-1-\frac{1}{y^2}.$$

Taking derivative w.r.t. x again,

$$y''=2y^{-3}y'=2y^{-3}\left(-\frac{1+y^2}{y^2}\right)=-\frac{2(1+y^2)}{y^5}.$$

4

2). Since both the base and the exponential have the variable x, we need to take the natural logarithm first.

$$\ln y = x(\ln x - \ln(1+x)).$$

Taking derivative with respect to x, we obtain

$$\frac{1}{y}y' = (\ln x - \ln(1+x)) + x(\frac{1}{x} - \frac{1}{1+x}).$$

So

$$y' = y\left(\ln\frac{x}{1+x} + \frac{1}{1+x}\right).$$

Taking derivative with respect to x again, we have

$$\begin{aligned} y'' &= y'\left(\ln\frac{x}{1+x} + \frac{1}{1+x}\right) + y\left(\frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2}\right) \\ &= y\left(\ln\frac{x}{1+x} + \frac{1}{1+x}\right)^2 + y\left(\frac{1}{x(1+x)^2}\right) \\ &= \left(\frac{x}{1+x}\right)^x\left(\left(\ln\frac{x}{1+x}\right)^2 + \frac{2}{1+x}\ln\frac{x}{1+x} + \frac{1}{x(1+x)}\right). \end{aligned}$$

3). This is the parameter equation with parameter t. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'_t}{x'_t} = \frac{f'(t) + tf''(t) - f'(t)}{f''(t)} = t.$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)'_t}{x'_t} = \frac{t'}{f''(t)} = \frac{1}{f''(t)}.$$

∎

Problem 5. Find the maximal value of the sequence $\{\sqrt[n]{n}\}$ for positive integer n.
Keywords. Maximal of a function, logarithmic differentiation.
Suggested Solution. Let $f(x) = x^{\frac{1}{x}}$ where $(x > 0)$. Then $\ln f(x) = \frac{1}{x}\ln x$. So

$$f'(x) = f(x)(\frac{1}{x}\ln x)' = f(x)\frac{1-\ln x}{x^2} = x^{\frac{1}{x}-2}(1-\ln x).$$

Let $f'(x) = 0$. We have $x = \mathrm{e}$. When $0 < x < \mathrm{e}$, $f'(x) > 0$. So $f(x)$ is strictly increasing in the interval $(0, \mathrm{e}]$, and hence $f(1) < f(2)$. When $\mathrm{e} < x$, $f'(x) < 0$. So $f(x)$ is strictly decreasing in the interval (e, ∞), and hence for any integer $n \geq 3$, we have $f(n) \leq f(3)$. Since $f(2)^6 = 2^3 < 3^2 = f(3)^6$, then $f(2) < f(3)$. So $\max\{\sqrt[n]{n}\} = \max\{f(n)\} = f(3) = \sqrt[3]{3}$.

∎

Problem 6. Compute the following indefinite integral:

$$1).\ \int x\ln(x-1)\,\mathrm{d}x. \qquad 2).\ \int \mathrm{e}^{-x}\cos x\,\mathrm{d}x.$$

Keywords. Integral by parts.
Suggested Solution.

1).

$$\begin{aligned}\int x\ln(x-1)\,\mathrm{d}x &= \frac{1}{2}\int \ln(x-1)\,\mathrm{d}x^2\\ &= \frac{1}{2}\left(x^2\ln(x-1)-\int\frac{x^2}{x-1}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(x^2\ln(x-1)-\int\frac{x^2-1+1}{x-1}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(x^2\ln(x-1)-\int(x+1)\,\mathrm{d}x-\int\frac{1}{x-1}\,\mathrm{d}(x-1)\right)\\ &= \frac{1}{2}x^2\ln(x-1)-\frac{1}{4}(x+1)^2-\frac{1}{2}\ln(x-1)+\mathrm{C}.\end{aligned}$$

2).

$$\begin{aligned}\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x &= \int \mathrm{e}^{-x}\,\mathrm{d}(\sin x)\\ &= \mathrm{e}^{-x}\sin x+\int \mathrm{e}^{-x}\sin x\,\mathrm{d}x\\ &= \mathrm{e}^{-x}\sin x-\int \mathrm{e}^{-x}\,\mathrm{d}(\cos x)\\ &= \mathrm{e}^{-x}\sin x-\mathrm{e}^{-x}\cos x-\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x.\end{aligned}$$

So

$$\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x=\frac{\mathrm{e}^{-x}}{2}(\sin x-\cos x)+\mathrm{C}.$$

∎

Problem 7.

1). Calculate $\int\sec\theta\,\mathrm{d}\theta$. (Hint: let $u=\sec\theta+\tan\theta$.)

2). Calculate $\int\frac{\mathrm{d}x}{\sqrt{x^2-1}}$, $x>1$ by substituting $x=\sec\theta$.

3). Recall $\sinh(t)=\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}$, $\cosh(t)=\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}$. Let $x=\cosh(t)$, it is obvious that x is an even function of t, so the usual inverse function does not exist. However, we can restrict to $t\ge 0$. Find the inverse function of $x=\cosh(t)$ for $t\ge 0$.

4). Calculate $\int\frac{\mathrm{d}x}{\sqrt{x^2-1}}$, $x>1$ by substituting $x=\cosh t$.

Keywords. Using the substitution to find integral, hyper-trigonometric functions.
Suggested Solution.

1) **1st method**. By the hint, setting $u=\sec\theta+\tan\theta$, we find

$$\mathrm{d}u=(\sec\theta\tan\theta+\sec^2\theta)\,\mathrm{d}\theta=u\sec\theta\,\mathrm{d}\theta.$$

So

$$\begin{aligned}\int \sec\theta\, d\theta &= \int \frac{u \sec\theta\, d\theta}{u} \\ &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

2nd method.

$$\begin{aligned}\int \sec\theta\, d\theta &= \int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta\, d\theta}{\cos^2\theta} = \int \frac{d\sin\theta}{1-\sin^2\theta} \\ &= \int \frac{dt}{(1-t)(1+t)} \quad \text{let } t = \sin\theta \\ &= \frac{1}{2}\int \frac{1}{1-t} + \frac{1}{1+t}\, dt = \frac{1}{2}(-\ln|1-t| + \ln|1+t|) + C \\ &= \ln\sqrt{\frac{|1+t|}{|1-t|}} + C = \ln\sqrt{\frac{|1+t|^2}{|(1-t)(1+t)|}} + C \\ &= \ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + C = \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

2). Let $x = \sec\theta$ for $0 < \theta < \frac{\pi}{2}$. Then

$$dx = \sec\theta \tan\theta\, d\theta.$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sec\theta \tan\theta\, d\theta}{\tan\theta} = \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|x + \sqrt{x^2-1}| + C.\end{aligned}$$

Since $(x+\sqrt{x^2-1})(x-\sqrt{x^2-1}) = 1$ and $x+\sqrt{x^2-1} \geq x-\sqrt{x^2-1} > 0$, we have $x+\sqrt{x^2-1} > 1$. So

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln(x+\sqrt{x^2-1}) + C \quad \text{for } x > 1.$$

3). Let $x = \cosh t = \frac{e^t+e^{-t}}{2}$ where $t \geq 0$. Then

$$(e^t)^2 - 2x(e^t) + 1 = 0.$$

Since $t \geq 0$, we have $e^t \geq 1$. So $e^t = x + \sqrt{x^2-1}$, where the other root is dropped because it is smaller than 1. Hence $t = \cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$.

4). Let $x = \cosh t > 1$. So $dx = \sinh t\, dt$. Since

$$\cosh^2 t - \sinh^2 t = 1,$$

we find

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} &= \int \frac{\sinh t\ \mathrm{d}t}{\sinh t} = \int \mathrm{d}t = t + \mathrm{C} \\ &= \ln(x + \sqrt{x^2-1}) + \mathrm{C}.\end{aligned}$$

∎

Problem 8. Consider the improper integral $\int_2^{+\infty} \frac{\mathrm{d}x}{x(\ln x)^p}$.

1). For what values of p is the improper integral divergent?

2). For what values of p is the improper integral convergent?

3). In the case of convergence, for what value of p is the improper integral minimal? Find the minimal value of the improper integral.

4). (**Logarithmic p-series**) For what values of p is the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

convergent?

Keywords. Improper integral, convergent, divergent, minimal value.
Suggested Solution. Let $A > 2$.

$$\begin{aligned}\int_2^A \frac{\mathrm{d}x}{x(\ln x)^p} &= \int_2^A \frac{\mathrm{d}(\ln x)}{(\ln x)^p} \\ &= \begin{cases} \ln\ln A - \ln\ln 2 & p = 1; \\ \frac{1}{1-p}\left((\ln A)^{1-p} - (\ln 2)^{1-p}\right) & p \neq 0. \end{cases}\end{aligned}$$

1). When $p = 1$,

$$\int_2^{+\infty} \frac{\mathrm{d}x}{x(\ln x)^p} = \lim_{A\to+\infty} (\ln\ln A - \ln\ln 2) = +\infty.$$

When $p < 1$,

$$\int_2^{+\infty} \frac{\mathrm{d}x}{x(\ln x)^p} = \lim_{A\to+\infty} \left(\frac{1}{1-p}\left((\ln A)^{1-p} - (\ln 2)^{1-p}\right)\right) = +\infty.$$

So the improper integral is divergent if $p \leq 1$.

2). When $p > 1$,

$$\int_2^{+\infty} \frac{\mathrm{d}x}{x(\ln x)^p} = \lim_{A\to+\infty} \left(\frac{1}{1-p}\left((\ln A)^{1-p} - (\ln 2)^{1-p}\right)\right) = \frac{(\ln 2)^{1-p}}{p-1}.$$

3). For $p > 1$, let $f(p) = \frac{(\ln 2)^{1-p}}{p-1}$. So

$$\begin{aligned} f'(p) &= \frac{(\ln 2)^{1-p}\ln(\ln 2)(-1)(p-1) - (\ln 2)^{1-p}}{(p-1)^2} \\ &= \frac{(\ln 2)^{1-p}(-\ln(\ln 2))\left(p - (1 - \frac{1}{\ln\ln 2})\right)}{(p-1)^2}. \end{aligned}$$

Since $2 < \mathrm{e}$, we know $\ln 2 < \ln \mathrm{e} = 1$. So $\ln\ln 2 < \ln 1 = 0$. Let $p_0 = 1 - \frac{1}{\ln\ln 2}$. Then $p_0 > 1$. When $1 < p < p_0$, we obtain $f'(p) < 0$. When $p > p_0$, we find $f'(p) > 0$. So when $p = p_0$, the improper integral is minimal. The minimal value is

$$f(p_0) = \frac{(\ln 2)^{\frac{1}{\ln\ln 2}}}{-\frac{1}{\ln\ln 2}} = -(\ln\ln 2)(\ln 2)^{\frac{1}{\ln\ln 2}}.$$

4). By the integral test for the convergence of series, when $p > 1$, the logarithmic p-series converges.

∎

Remark. Let us recall the integral test. Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = g(n)$, where g is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} g(x)\,\mathrm{d}x$ both converge or both diverge.

Problem 9. Consider the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^p}$.

1). For what values of p is the series divergent?

2). For what values of p is the series absolutely convergent?

3). For what values of p is the series conditionally convergent?

Keywords. Alternating series, divergence, absolutely convergence, conditional convergence, Leibniz's theorem for the alternating series test.
Suggested Solution. Let $a_n = (-1)^n \frac{1}{n^p}$.

1). When $p \leq 0$, we find $\lim_{n\to\infty} a_n \neq 0$. So the alternating series diverges.

2). We have

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The p-series is convergent for $p > 1$ (by the integral test). So when $p > 1$, the alternating series is absolutely convergent.

3). When $0 < p \leq 1$, the alternating series satisfies the assumption of the Leibniz's theorem and conditionally converges.

$\blacksquare$

Remark. Let us recall the Leibniz's theorem. The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges if all three of the following conditions are satisfied.

1). The a_n's are all positive.

2). $a_n \geq a_{n+1}$ for $n \geq N$, for some integer N.

3). $a_n \to 0$.

The theorem is proved by the trick of pairing: let $b_1 = (a_1 - a_2)$, $b_2 = (a_3 - a_4)$, $\cdots$ Then the alternating sum of a_n will become the positive sum of b_n.

Problem 10. Let $h \in (0, \pi)$. Consider the function

$$f(x) = \begin{cases} 1 & 0 \leq x \leq h; \\ 0 & h < x \leq \pi. \end{cases}$$

1). Find the Fourier sine series for $f(x)$. What is the value of the Fourier sine at the discontinuity point $x = h$?

2). Find the Fourier cosine series for $f(x)$. What is the value of the Fourier cosine at the discontinuity point $x = h$?

Keywords. Fourier cosine series, Fourier sine series.
Suggested Solution.

1). Regardless the discontinuity points, we define the odd extension of f by

$$F(x) = \begin{cases} 1 & 0 < x < h; \\ 0 & h < x < \pi; \\ -1 & -h < x < 0; \\ 0 & -\pi < x < -h. \end{cases}$$

Then we apply a periodic extension with period 2π to $F(x)$, and still denote it by $F(x)$. The Fourier coefficients are

$$\begin{aligned} a_n &= 0, \quad n = 0, 1, 2, \cdots \\ b_n &= \frac{2}{\pi} \int_0^{\pi} F(x) \sin nx \, \mathrm{d}x = \frac{2}{\pi} \int_0^{h} \sin nx \, \mathrm{d}x = \frac{2(1 - \cos nh)}{n\pi} \qquad n = 1, 2, \cdots. \end{aligned}$$

So the Fourier sine of $F(x)$ is

$$g(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos nh}{n} \sin nx.$$

Applying the convergence theorem of Fourier series, we have

$$f(x) = g(x) \quad \text{for } x \in (0,h) \cup (h,\pi).$$

At the jump discontinuity point $x = h$, the Fourier series converges to the average

$$g(h) = \frac{f(h^+) + f(h^-)}{2} = \frac{1}{2}.$$

2). Regardless the discontinuity points, we define the even extension of f by

$$F(x) = \begin{cases} 1 & 0 < x < h; \\ 0 & h < x < \pi; \\ 1 & -h < x < 0; \\ 0 & -\pi < x < -h. \end{cases}$$

Then we apply a periodic extension with period 2π to $F(x)$, and still denote it by $F(x)$. The Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi F(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^h \mathrm{d}x = \frac{2h}{\pi}, \\ a_n &= \frac{2}{\pi}\int_0^\pi F(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^h \cos nx\,\mathrm{d}x = \frac{2\sin nh}{n\pi} \quad n = 1,2,\cdots; \\ b_n &= 0, \quad n = 1,2,\cdots. \end{aligned}$$

So the Fourier cosine of $F(x)$ is

$$g(x) = \frac{h}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin nh}{n}\cos nx.$$

Applying the convergence theorem of Fourier series, we have

$$f(x) = g(x) \quad \text{for } x \in (0,h) \cup (h,\pi).$$

At the jump discontinuity point $x = h$, the Fourier series converges to the average

$$g(h) = \frac{f(h^+) + f(h^-)}{2} = \frac{1}{2}.$$

∎

Remark The Fourier series of a function $f(x)$ defined on the interval $-L < x < L$ is

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$.
The convergence theorem of Fourier series states that:
If the function f and its derivative f' are piecewise continuous over the interval $-L < x < L$, then $f(x)$ equals its Fourier series $g(x)$ at all points of continuity. At a point h where a jump discontinuity happens in f, the Fourier series converges to the average

$$g(h) = \frac{f(h^+) + f(h^-)}{2},$$

where $f(h^+)$ and $f(h^-)$ means the right and the left limits of f at h, respectively.

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Calculus II

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This is a tutorial demo training to all new Teaching Assistants. It contains ten typical problems in multi-variable Calculus. The new TA would be randomly assigned a problem, and present the problem on blackboard to other new TAs. Pros and Cons of each presentation would be highlighted and discussed after the presentation. Typical teaching techniques/tricks/mistakes would also be emphasised.

Problem 1. Determine whether the following limits exist. If they do exist then find their value.

$$1).\ \lim_{(x,y)\to(0,0)} \frac{\sin xy}{y}. \qquad 2).\ \lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}.$$

$$3).\ \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}. \qquad 4).\ \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$

Keywords. Multi-variable limit.
Suggested Solution.

1).

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} x = 1\cdot 0 = 0.$$

2).

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2} &= \lim_{(x,y)\to(0,0)} \frac{2\sin^2\frac{x^2+y^2}{2}}{(\frac{x^2+y^2}{2})^2}\frac{x^2+y^2}{4x^2y^2} \quad (1-\cos\theta = 2\sin^2\frac{\theta}{2})\\ &= \frac{1}{2}\cdot 1^2 \lim_{(x,y)\to(0,0)}\left(\frac{1}{x^2}+\frac{1}{y^2}\right) = +\infty.\end{aligned}$$

The limit does not exist.

3). We take two ways to compute the limit. The first way is taking $y = x$. The second way is taking $y = 2x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0}\frac{x^4}{x^4} = 1.$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=2x}} \frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0}\frac{4x^4}{4x^4+x^2} = \lim_{x\to 0}\frac{4x^2}{4x^2+1} = 0.$$

These two ways of computing are not consistent so the original two-variable limit does not exist.

4). We apply the basic inequality $|xy| \le \frac{x^2+y^2}{2}$. So

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{x^2+y^2}{2\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{2}.$$

For any $\varepsilon > 0$, taking $\delta = 2\varepsilon$, when $0 < \sqrt{x^2+y^2} < \delta$, we have

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| \le \frac{\sqrt{x^2+y^2}}{2} < \frac{\delta}{2} = \varepsilon.$$

By the definition of limit, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.

∎

Problem 2. Let $z(x, y) = \frac{y}{x^2+y^2}$.

1). Find Δz at the point $(x, y) = (3, 4)$ with $\Delta x = 0.1$, $\Delta y = 0.1$.

2). Compute dz.

3). Compute dz at the point $(x, y) = (3, 4)$ with $\Delta x = 0.1$, $\Delta y = 0.1$. What is the difference between Δz and dz at this time?

Keywords. total differential, change of function, partial derivative.
Suggested Solution.

1). By definition $\Delta z = z(x+\Delta x, y+\Delta y) - z(x,y)$. Putting the given data inside, we obtain

$$\Delta z\Big|_{\substack{(x,y)=(3,4)\\(\Delta x,\Delta y)=(0.1,0.1)}} = z(3.1,4.1) - z(3,4) = \frac{4.1}{26.42} - \frac{4}{25} \approx -0.00481.$$

2).

$$\begin{aligned} \mathrm{d}z &= \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = \frac{-y\cdot 2x}{(x^2+y^2)^2}\,\mathrm{d}x + \frac{(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2}\,\mathrm{d}y \\ &= \frac{-2xy}{(x^2+y^2)^2}\,\mathrm{d}x + \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}y. \end{aligned}$$

3). By the definition, we know $\mathrm{d}x = \Delta x$, $\mathrm{d}y = \Delta y$. So

$$\mathrm{d}z\Big|_{\substack{(x,y)=(3,4)\\(\Delta x,\Delta y)=(0.1,0.1)}} = \frac{-24}{25^2}\cdot 0.1 + \frac{-7}{(25)^2}\cdot 0.1 = -\frac{3.1}{25^2} = -0.00496.$$

We find that

$$\Delta z - \mathrm{d}z\Big|_{\substack{(x,y)=(3,4)\\(\Delta x,\Delta y)=(0.1,0.1)}} \approx -0.00481 - (-0.00496) = 0.00015,$$

which is higher order smaller than the size of the change of variable 0.1.

∎

Remark. In general we have an estimation on the error in the standard linear approximation. If $z = z(x,y)$ has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|z_{xx}|$, $|z_{xy}|$, and $|z_{yy}|$ on R, then the error $E(x,y) = z(x,y) - \mathrm{L}(x,y)$ of $z(x,y)$ and its standard linear approximation

$$\mathrm{L}(x,y) = z(x_0,y_0) + z'_x(x-x_0) + z'_y(y-y_0)$$

satisfies the inequality

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2.$$

So

$$\Delta z - \mathrm{d}z = E(x,y), \quad |\Delta z - \mathrm{d}z| \le \frac{1}{2}M(|\Delta x| + |\Delta y|)^2.$$

$\Delta z - \mathrm{d}z$ is higher order smaller than Δx or Δy.

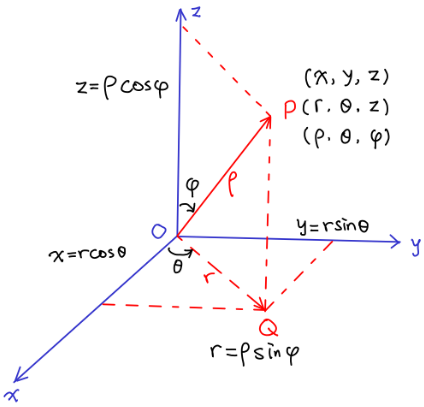
Problem 3. Define the 3-dimensional Laplace operator as $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, where (x,y,z) are variables in the Cartesian coordinates in $\mathbb{R}^3$.

1). Express the Laplace operator in cylinder coordinates.

2). Express the Laplace operator in spherical coordinates. (This part is optional.)

Keywords. The Laplace operator, cylinder coordinates, spherical coordinates, change of variable, chain rule.
Suggested Solution.

Let us recall that the change of variables form the Cartesian coordinate to the cylinder coordinates and to the spherical coordinates. We have the relations



$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$
$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x},$$

where $\theta \in [0, 2\pi)$, and the relations

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi,$$
$$\rho = \sqrt{x^2+y^2+z^2}, \quad r = \rho\sin\phi,$$

where $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi)$.

1). Let us compute the first order partial derivatives.

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2}2x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2} = -\frac{y}{r^2}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{r^2}.$$

Caution. We notice that $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$. We also know $x = r\cos\theta$. So $\frac{\partial x}{\partial r} = \cos\theta$. Therefore

$$\frac{\partial r}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial r}}.$$

Let us compute the second order partial derivatives.

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} + x\frac{\partial \frac{1}{r}}{\partial x} = \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x} = \frac{1}{r} - \frac{x^2}{r^3}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{y^2}{r^3}.$$

$$\frac{\partial^2 \theta}{\partial x^2} = -y\frac{\partial \frac{1}{r^2}}{\partial x} = \frac{2y}{r^3}\frac{x}{r} = \frac{2xy}{r^4}, \quad \frac{\partial^2 \theta}{\partial y^2} = x\frac{\partial \frac{1}{r^2}}{\partial x} = -\frac{2x}{r^3}\frac{y}{r} = -\frac{2xy}{r^4}.$$

We find that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y},$$

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
\frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{\partial}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}.
\end{aligned}$$

Hence

$$
\begin{aligned}
\triangle &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\\
&= \frac{\partial^2}{\partial r^2}\frac{x^2+y^2}{r^2}+\frac{\partial}{\partial r}\left(\frac{2}{r}-\frac{x^2+y^2}{r^3}\right)+\frac{\partial^2}{\partial \theta^2}\frac{x^2+y^2}{r^4}+\frac{\partial}{\partial \theta}\frac{2xy-2xy}{r^4}+\frac{\partial^2}{\partial z^2}\\
&= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2} \qquad \text{(also written as below)}\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}.
\end{aligned}
$$

2). We change the cylinder coordinates to spherical coordinates. The relations are

$$
r=\rho\sin\phi,\quad \theta=\theta,\quad z=\rho\cos\phi;
$$
$$
\rho=\sqrt{r^2+z^2},\quad \theta=\theta,\quad \phi=\arctan\frac{r}{z}.
$$

Let us compute the first order partial derivatives.

$$
\frac{\partial\rho}{\partial r}=\frac{r}{\rho}=\sin\phi,\quad \frac{\partial\rho}{\partial z}=\cos\phi,\quad \frac{\partial\phi}{\partial r}=\frac{1}{1+\left(\frac{r}{z}\right)^2}\frac{1}{z}=\frac{\cos\phi}{\rho},\quad \frac{\partial\phi}{\partial z}=-\frac{\sin\phi}{\rho}.
$$

Let us compute the second order partial derivatives.

$$
\frac{\partial^2\rho}{\partial r^2}=\cos\phi\frac{\partial\phi}{\partial r}=\frac{\cos^2\phi}{\rho},\quad \frac{\partial^2\rho}{\partial z^2}=-\sin\phi\frac{\partial\phi}{\partial z}=\frac{\sin^2\phi}{\rho};
$$

$$
\frac{\partial^2\phi}{\partial r^2}=-\frac{\sin\phi}{\rho}\frac{\partial\phi}{\partial r}-\frac{\cos\phi}{\rho^2}\frac{\partial\rho}{\partial r}=-\frac{2\sin\phi\cos\phi}{\rho^2},\quad \frac{\partial^2\phi}{\partial z^2}=\frac{2\sin\phi\cos\phi}{\rho^2}.
$$

We have

$$
\begin{aligned}
\frac{\partial}{\partial r}&=\frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial r}+\frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial r}=\sin\phi\frac{\partial}{\partial\rho}+\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi},\\
\frac{\partial}{\partial z}&=\frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial z}+\frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial z}=\cos\phi\frac{\partial}{\partial\rho}-\frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}.
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial^2}{\partial r^2}&=\sin^2\phi\frac{\partial^2}{\partial\rho^2}+\frac{\cos^2\phi}{\rho}\frac{\partial}{\partial\rho}+\frac{\cos^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2}-\frac{2\sin\phi\cos\phi}{\rho^2},\\
\frac{\partial^2}{\partial z^2}&=\cos^2\phi\frac{\partial^2}{\partial\rho^2}+\frac{\sin^2\phi}{\rho}\frac{\partial}{\partial\rho}+\frac{\sin^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{2\sin\phi\cos\phi}{\rho^2}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
\triangle &= \frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad \text{(cylinder coordinate)}\\
&= \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{1}{\rho\sin\phi}\left(\sin\phi\frac{\partial}{\partial\rho}+\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right)+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\\
&= \frac{\partial^2}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{1}{\rho\sin\phi}\left(\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right)+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} \qquad \text{(also written as below)}\\
&= \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}.
\end{aligned}$$

∎

Problem 4. Fix the ellipsoid $x^2+2y^2+z^2=1$.

1). Find the tangent planes on the ellipsoid which are parallel to the plane $x-y+2z=\sqrt{22}$.

2). Find the distance between the two surfaces. The distance dist between two surfaces α, β is defined as: $\text{dist}(\alpha,\beta)=\min\{\text{dist}(P,Q)|\ P\in\alpha,\ Q\in\beta\}$.

Keywords. Tangent plane, normal vector, distance.
Suggested Solution.

1). Let $F(x,y,z)=x^2+2y^2+z^2-1$. Then

$$\boldsymbol{n}=(\boldsymbol{F}_x,\boldsymbol{F}_y,\boldsymbol{F}_z)=(2x,4y,2z).$$

The normal vector of the plane is $\boldsymbol{a}=(1,-1,2)$. Since this plane is parallel to the tangent planes, we obtain $\boldsymbol{n}\parallel\boldsymbol{a}$, i.e.,

$$\frac{2x}{1}=\frac{4y}{-1}=\frac{2z}{2}.$$

So we find $x=\frac{1}{2}z$, $y=-\frac{1}{4}z$. Substituting them into ellipsoid, we have

$$\left(\frac{z}{2}\right)^2+2\left(-\frac{z}{4}\right)^2+z^2=1.$$

Then $z=\pm2\sqrt{\frac{2}{11}}$, $x=\pm\sqrt{\frac{2}{11}}$, $y=\mp\frac{1}{2}\sqrt{\frac{2}{11}}$. The tangent points are

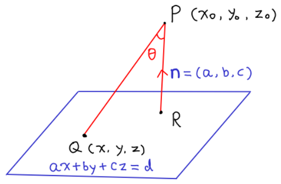
$$P_1=\left(\sqrt{\frac{2}{11}},-\frac{1}{2}\sqrt{\frac{2}{11}},2\sqrt{\frac{2}{11}}\right),\quad P_2=\left(-\sqrt{\frac{2}{11}},\frac{1}{2}\sqrt{\frac{2}{11}},-2\sqrt{\frac{2}{11}}\right).$$

The tangent planes are

$$\left(x\pm\sqrt{\frac{2}{11}}\right)-\left(y\mp\frac{1}{2}\sqrt{\frac{2}{11}}\right)+2\left(z\pm2\sqrt{\frac{2}{11}}\right)=0,$$

i.e., $x-y+2z=\pm\frac{\sqrt{22}}{2}$

2). The distance of the two surfaces is the distance of P_1 to the original plane. Let $P = (x_0, y_0, z_0)$ be a point outside the plane $\beta: ax + by + cz = d$. Let $Q = (x, y, z)$ be any point on the plane. Let R be the projection of P to β. We see that $\boldsymbol{n} = (a, b, c)$ is a normal vector of the plane.



$$
\begin{aligned}
\mathrm{dist}(P, \beta) &= |PR| = |PQ||\cos\theta| = \left|\overrightarrow{PQ} \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right| \\
&= \frac{\left|(x - x_0, y - y_0, z - z_0) \cdot (a, b, c)\right|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{\left|a(x - x_0) + b(y - y_0) + c(z - z_0)\right|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{\left|ax_0 + by_0 + cz_0 - d)\right|}{\sqrt{a^2 + b^2 + c^2}},
\end{aligned}
$$

where in the last equation, we use the condition $Q \in \beta$. Now

$$
\mathrm{dist}(\alpha, \beta) = \mathrm{dist}(P_1, \beta) = \frac{|\frac{\sqrt{22}}{2} - \sqrt{22}|}{\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{\sqrt{33}}{6}.
$$

∎

Problem 5. Let S be the sphere $x^2 + y^2 + z^2 = a^2$ and C be the cylinder $x^2 + y^2 = ax$ (a>0).

1). Compute the area of the part of the sphere inside the cylinder.

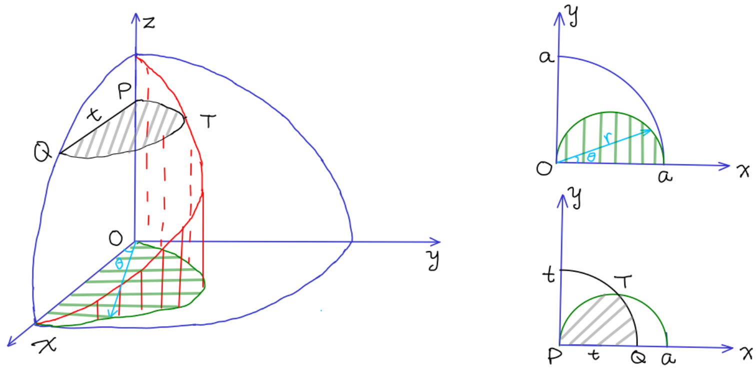
2). Write down the volume of the region Ω bounded by the cylinder in terms of triple integral. Explicitly express the triple integral in three repeated single integrations (no need to compute the final integration).

Keywords. Double integration, triple integration.
Suggested Solution.

1). By the symmetry, we only need to computer in the first octant. In this case the sphere is given by the function $z = \sqrt{a^2 - x^2 - y^2}$. Then $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}$, $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$. Let D

be the projection of the inside parts of the sphere onto the xy plane. We find

$$\begin{aligned}
\text{Area} &= 4\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y = 4\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y \\
&= 4a\iint_D \frac{1}{a^2-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta = 4a\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{a\theta}\frac{1}{\sqrt{a^2-r^2}}\,r\,\mathrm{d}r \\
&= 2a\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\frac{-1}{\sqrt{a^2-r^2}}\,\mathrm{d}(a^2-r^2) = 2a\int_0^{\frac{\pi}{2}}\left[-2(a^2-r^2)^{\frac{1}{2}}\right]_0^{a\cos\theta}\mathrm{d}\theta \\
&= 4a\int_0^{\frac{\pi}{2}} a(1-\sin\theta)\,\mathrm{d}\theta = 2a^2(\pi-2).
\end{aligned}$$



2). We sketch the region Ω. Let $P=(0,0,z)$. The intersection of the horizontal plane passing through P and the region Ω is sketching as above picture, which is bounded by the circle from P to T, and circle from T to Q. The point Q is given by $(t,0,0)$. So the coordinate of T can be computed as the following equations.

$$\begin{cases} x^2+y^2 &= t^2 \\ x^2+y^2 &= ax \\ z^2+t^2 &= a^2 \end{cases} \Longrightarrow (x,y,z) = \left(\frac{t^2}{a}, \frac{t\sqrt{a^2-t^2}}{a}, \sqrt{a^2-t^2}\right).$$

So the volume of Ω:

$$\text{Vol}(\Omega) = 4\iiint_\Omega \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 4\int_0^a \mathrm{d}z\int_0^{\frac{t\sqrt{a^2-t^2}}{a}}\mathrm{d}y\int_{\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-y^2}}^{\sqrt{t^2-y^2}}\mathrm{d}x.$$

∎

Problem 6. Let Ω be the region bounded by the cylinder $x^2+y^2=1$, and planes $z=1$, $z=0$, $x=1$, $y=0$ in the first octant.

1). Compute $\iiint\limits_{\Omega} xy\,\mathrm{d}V$ by using the cylinder coordinate.

2). Compute $\iiint\limits_{\Omega} xy\,\mathrm{d}V$ by using the Cartesian coordinate.

3). Compute $\iiint\limits_{\Omega} xy\,\mathrm{d}V$ by using Gauss' divergence theorem.

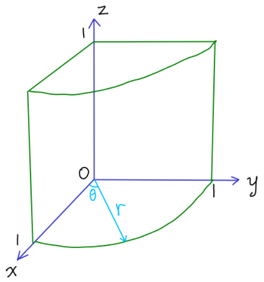
Keywords. Triple integration, the cylinder coordinate, Gauss' divergence theorem.
Suggested Solution.

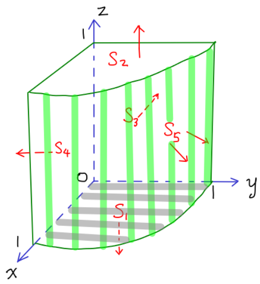
1).

$$\begin{aligned}\iiint\limits_{\Omega} xy\,\mathrm{d}V &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^1 r\,\mathrm{d}r \int_0^1 r^2 \sin\theta\cos\theta\,\mathrm{d}z \\ &= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin 2\theta\,\mathrm{d}(2\theta)\int_0^1 r^3\,\mathrm{d}r = \frac{1}{16}(-\cos 2\theta)\Big|_0^{\frac{\pi}{2}} = \frac{1}{8}.\end{aligned}$$

2).

$$\begin{aligned}\iiint\limits_{\Omega} xy\,\mathrm{d}V &= \int_0^1 x\,\mathrm{d}x \int_0^{\sqrt{1-x^2}} y\,\mathrm{d}y \int_0^1 \mathrm{d}z = \int_0^1 x\,\mathrm{d}x \int_0^{\sqrt{1-x^2}} y\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^1 x(1-x^2)\,\mathrm{d}x = \frac{1}{4}\int_0^1 (1-t)\,\mathrm{d}t = \frac{1}{8}.\end{aligned}$$





3). Gauss' divergence theorem states that

$$\iiint\limits_{\Omega} \operatorname{div}\mathbf{F}\,\mathrm{d}V = \oiint\limits_{S} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\sigma,$$

where $\mathbf{F}$ is a smooth vector field on Ω, $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$ is the divergence of $\mathbf{F}$, S is the boundary of Ω, $\mathbf{n}$ is the outward normal vector of S. To use this theorem for the problem, we need to construct a vector field on V. Let us define $\mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + xyz\mathbf{k} = xyz\mathbf{k}$. Then $\operatorname{div}\mathbf{F} = \frac{\partial xyz}{\partial z} = xy$. We split the surface into $S_1, \cdots, S_5$. Denote $\mathbf{n}_i$ the outward normal vector of S_i. Then

$$\mathbf{F}\cdot\mathbf{n}_i = 0, \quad i = 3,4,5.$$

Notice that $\mathbf{F} = (0,0,0)$ on S_1. We have

$$\begin{aligned}
\iiint\limits_{\Omega} xy\,\mathrm{d}V &= \iiint\limits_{\Omega} \mathrm{div}\,\mathbf{F}\ \mathrm{d}V = \Sigma_{i=1}^{5} \oiint\limits_{S_i} \mathbf{F}\cdot\mathbf{n_i}\,\mathrm{d}\sigma \\
&= \oiint\limits_{S_2} (0,0,xy)\cdot(0,0,1)\,\mathrm{d}\sigma = \iint\limits_{\substack{x^2+y^2\le 1\\ x\ge 0,y\ge 0}} xy\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\int_0^{\frac{\pi}{2}} r^3 \sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{8}.
\end{aligned}$$

∎

Problem 7. Let Γ be the intersection of the plane $y = z$ and the sphere $x^2 + y^2 + z^2 = 1$, in the direction counterclockwise with respect to the z-axis.

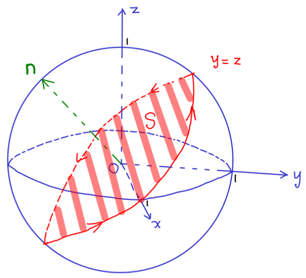
1). Compute the integral $\oint\limits_{\Gamma} xyz\,\mathrm{d}z$ by parametrization the curve Γ.

2). Compute the integral $\oint\limits_{\Gamma} xyz\,\mathrm{d}z$ by Stocks' theorem.

Keywords. Line integral, orientation, Stocks' theorem.
Suggested Solution.

1). The intersection of plane and the unit sphere is the following.

$$\begin{cases} x^2+y^2+z^2=1 \\ y=z \end{cases} \Longrightarrow \begin{cases} x^2+2z^2=1 \\ y=z \end{cases}$$



We take the parameter equations,

$$\begin{cases} x = \cos t \\ y = \frac{\sqrt{2}}{2}\sin t \\ z = \frac{\sqrt{2}}{2}\sin t \end{cases} \qquad t: 0 \to 2\pi.$$

$$\begin{aligned}
\oint_\Gamma xyz\,\mathrm{d}z &= \int_0^{2\pi} \cos t\left(\frac{\sqrt{2}}{2}\sin t\right)\left(\frac{\sqrt{2}}{2}\sin t\right)\left(\frac{\sqrt{2}}{2}\cos t\right)\mathrm{d}t \\
&= \frac{\sqrt{2}}{4}\int_0^{2\pi} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{\sqrt{2}}{16}\int_0^{2\pi}\sin^2(2t)\,\mathrm{d}t \\
&= \frac{\sqrt{2}}{16}\int_0^{2\pi}\frac{1-\cos 4t}{2}\,\mathrm{d}t = \frac{\sqrt{2}}{16}\pi.
\end{aligned}$$

2). The circulation of a vector field $\mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + xyz\mathbf{k}$ around the boundary Γ of the oriented disc S in the direction counterclockwise with respect to its normal vector $\mathbf{n} = (0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ equals the integral of $\nabla\times\mathbf{F}\cdot\mathbf{n}$ over S.

$$\oint_\Gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\sigma.$$

$$\oint_\Gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_\Gamma xyz\mathbf{k}\cdot(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z) = \oint_\Gamma xyz\,\mathrm{d}z.$$

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \nabla\times\mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & xyz \end{vmatrix} \\
&= \frac{\partial xyz}{\partial y}\mathbf{i} - \frac{\partial xyz}{\partial x}\mathbf{j} = xz\mathbf{i} - yz\mathbf{j} = (xz, -yz, 0).
\end{aligned}$$

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\sigma = \iint_S (xz,-yz,0)\cdot(0,-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})\,\mathrm{d}\sigma = \frac{\sqrt{2}}{2}\iint_S yz\,\mathrm{d}\sigma$$

Now S is in the plane $z = y$, so $\mathrm{d}\sigma = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{1+0^2+1^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y$. Then we find that

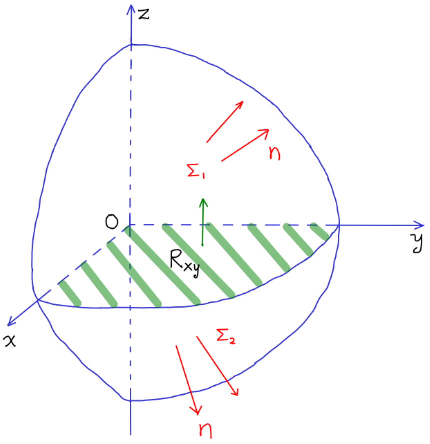
$$\frac{\sqrt{2}}{2}\iint_S yz\,\mathrm{d}\sigma = \iint_{x^2+2y^2=1} y^2\,\mathrm{d}x\,\mathrm{d}y.$$

Let $x = r\cos\theta$, $y = \frac{\sqrt{2}}{2}r\sin\theta$. We obtain $\mathrm{d}x\,\mathrm{d}y = \frac{\sqrt{2}}{2}r\,\mathrm{d}r\,\mathrm{d}\theta$, and hence

$$\begin{aligned}
\iint_{x^2+2y^2=1} y^2\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1\int_0^{2\pi}\frac{1}{2}r^2\sin^2\theta\left(\frac{\sqrt{2}}{2}\right)r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{\sqrt{2}}{4}\int_0^1 r^3\,\mathrm{d}r\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta = \frac{\sqrt{2}}{16}\pi.
\end{aligned}$$

■

Problem 8. Computer the surface integral $\iint_{\Sigma} xyz\,\mathrm{d}x\,\mathrm{d}y$, where Σ is the parts of unit sphere in the region $x \geq 0$, $y \geq 0$, with normal vector outward.
Keywords. Surface integral, oriented surface.
Suggested Solution. Let us sketch the graph.



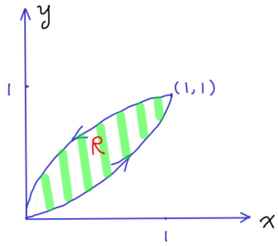
We divide Σ in two parts. Σ_1 is given by the equation $z = \sqrt{1-x^2-y^2}$, and Σ_2 is given by the equation $z = -\sqrt{1-x^2-y^2}$. The orientation of Σ_1 is the same as the xy-plane, and is upward. (The really meaning is that the dot product of the two normal vector for the two surfaces are positive.) The orientation of Σ_2 is the opposite to the xy-plane. Denote the projection of Σ_1 (or Σ_2) to the xy-plane by R_{xy}. Then we obtain

$$\begin{aligned}
\iint_{\Sigma} xyz\,\mathrm{d}x\,\mathrm{d}y &= \iint_{\Sigma_1} xyz\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Sigma_2} xyz\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{R_{xy}} xy\sqrt{1-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y - \iint_{R_{xy}} xy(-\sqrt{1-x^2-y^2})\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\iint_{R_{xy}} xy\sqrt{1-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\iint_{R_{xy}} r^2\sin\theta\cos\theta\sqrt{1-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta \qquad x = r\cos\theta,\ \ y = r\sin\theta \\
&= \int_0^{\pi/2}\sin 2\theta\,\mathrm{d}\theta\int_0^1 r^3\sqrt{1-r^2}\,\mathrm{d}r \\
&= \int_0^1 (1-s^2)s^2\,\mathrm{d}s \qquad s = \sqrt{1-r^2} \\
&= \frac{2}{15}.
\end{aligned}$$

■

Problem 9. Using Green's theorem to compute the integral $\oint_C (2xy - x^2 + x)\,\mathrm{d}x + (x + y^4 + y^2)\,\mathrm{d}y$, where C is the boundary of the region R cutting out by two parabolas $y = x^2$ and $x = y^2$, with counterclockwise orientation.
Keywords. Green's theorem in the plane.
Suggested Solution. Let us recall Green's theorem in flux-divergence form. The outward flux of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \oint_C M\,\mathrm{d}y - N\,\mathrm{d}x = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

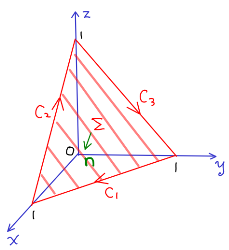


Now let us take $M = x + y^4 + y^2$, $N = x^2 - x - 2xy$. We have

$$\begin{aligned}
\oint_C (2xy - x^2 + x)\,\mathrm{d}x + (x + y^4 + y^2)\,\mathrm{d}y &= \iint_R (1-2x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{y^2}^{\sqrt{y}} (1-2x)\,\mathrm{d}x \\
&= \int_0^1 [x - x^2]_{y^2}^{\sqrt{y}}\,\mathrm{d}y = \int_0^1 (\sqrt{y} - y - y^2 + y^4)\,\mathrm{d}y \\
&= \frac{2}{3} - \frac{1}{2} - \frac{1}{3} + \frac{1}{5} = \frac{1}{30}.
\end{aligned}$$

∎

Problem 10. Finding the work done by $\mathbf{F} = y\mathbf{i} + z\mathbf{y} + x\mathbf{z}$ over the curve C, which is the intersection of the plane $x + y + z = 1$ with three coordinate planes, and clockwise orientation from z-axis.
Keywords. Work done by a variable force over a space curve, Stocks' theorem.
Suggested Solution.



Method 1). Let Σ be the region of the plane $x+y+z=1$ in the first octant, with norm vector $\mathbf{n} = -\frac{1}{\sqrt{3}}(1,1,1)$. Then the work

$$\begin{aligned}
W &= \oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\sigma \\
&= \iint_\Sigma \begin{vmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \mathrm{d}\sigma \\
&= \sqrt{3}\iint_\Sigma \mathrm{d}\sigma = \sqrt{3}\,\text{Area(Equilateral triangle)} \\
&= \sqrt{3}\times\frac{1}{2}(\sqrt{2})\times\sin\frac{\pi}{3} = \frac{3}{2}.
\end{aligned}$$

Method 2). Let us divide C in two three line segments C_1, C_2, C_3.

$$\begin{aligned}
W &= \oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_C y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z \\
&= \left(\int_{C_1} y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z\right) + \left(\int_{C_2} y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z\right) + \left(\int_{C_3} y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z\right) \\
&= 3\int_{C_1} y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z \qquad \text{by the symmetry of the functions and regions} \\
&= 3\int_{\substack{x+y=1,z=0\\0\le x\le 1}} y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z = 3\int_0^1 (1-x)\,\mathrm{d}x = \frac{3}{2}.
\end{aligned}$$

∎