ODE, SageMath, and General AI - An Example From High School Teaching

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Abstract

In this teaching notes, we solve an ordinary differential equation using mathematical logic. Additionally, we present a solution obtained through the mathematical software SageMath. We also test the ODE on various General AI systems, finding that they are unable to solve this particular ODE directly. This example highlights the rigor and beauty of mathematics, demonstrates the power of SageMath, and reveals the current limitations of general artificial intelligence in solving complex mathematical problems.

1 THE ODE PROBLEM AND SOLUTIONS.

We assume that the reader has basic knowledge of calculus in the level of high school, for example, as in the syllabus [1] for MATHEMATICS 3C and MATHEMATICS 5 from Swedish National Agency for Education.

The following ordinary differential equation problem is raised in a teaching in MATHEMATICS 5.

The ODE Problem. Solve the ordinary differential equation

$$y''(x) - 8y'(x) + 16y(x) = xe^{4x} \tag{1.1}$$

with the conditions $y'(0) = 1$ and $y'(1) = e^4$.

Solution. The equation (1.1) is a second order inhomogeneous linear ODE.

Step 1. Solutions of corresponding homogeneous equation. We solve

$$y''(x) - 8y'(x) + 16y(x) = 0. \tag{1.2}$$

Its characteristic equation is given by

$$r^2 - 8r + 16 = 0. \tag{1.3}$$

The solution of (1.3) is given by $r = 4$ with multiplicity 2. By the standard theory of ODE (see for example [2] or [3, Ch. 19]), the solution of (1.2) is given by

$$y_c(x) = e^{4x}(Ax + B), \tag{1.4}$$

This note is based on my teaching at Kärrtorps Gymnasium in the spring of 2024. The author thanks Mr. Herman Katea for sharing the question.

where A and B are constants.

Step 2. Find a particular solution of inhomogeneous equation (1.1). We observe that the right side of (1.1) is a solution for the homogeneous equation (1.2). This feature of the right side of (1.1) is what makes the equation intriguing, presenting challenges specifically for AI solvers.

A guess of a particular solution of the form

$$y_p(x) = k_1 x^2 e^{4x} + k_2 x e^{4x} + k_3 e^{4x} \tag{1.5}$$

will not work, because $k_2 x e^{4x}$ and $k_3 e^{4x}$ are already solutions of homogeneous equation, and one can check that $y_p(x) = k_1 x^2 e^{4x}$ is not a solution of (1.1).

We must use new idea. The method is to let the solution be of the form

$$y(x) = e^{4x} g(x), \tag{1.6}$$

and we can convert the original ODE to a new ODE of the function $g(x)$.

By the chain rule and (1.6), we have

$$y'(x) = e^{4x}\left(4g(x) + g'(x)\right), \tag{1.7}$$

$$y''(x) = e^{4x}\left(16g(x) + 8g'(x) + g''(x)\right). \tag{1.8}$$

Putting (1.8) and (1.7) into left side of (1.1), we obtain

$$\begin{aligned}
&y''(x) - 8y'(x) + 16y(x) \\
&= e^{4x}\left(16g(x) + 8g'(x) + g''(x) - 32g(x) - 8g'(x) + 16g(x)\right) \\
&= e^{4x} g''(x).
\end{aligned}$$

Thus, from the original ODE (1.1), we derive a new ODE of function $g(x)$:

$$g''(x) = x. \tag{1.9}$$

The general solution of (1.9) is given by

$$g(x) = \frac{x^3}{6} + c_1 x + c_2.$$

Therefore a particular solution of the inhomogeneous ODE (1.1) is given by (taking $c_1 = c_2 = 0$ and using (1.6))

$$y_p(x) = e^{4x}\frac{x^3}{6},$$

and the general solution of the (1.1) is of the form

$$y(x) = y_c(x) + y_p(x) = e^{4x}\left(\frac{x^3}{6} + Ax + B\right), \tag{1.10}$$

where A and B are some constants.

Step 3. Find the constants A and B. From (1.10), we get

$$y'(x) = e^{4x}\left(4\frac{x^3}{6} + 4Ax + 4B + \frac{x^2}{2} + A\right). \tag{1.11}$$

So $y'(0) = (4B + A)$, $y'(1) = e^4\left(\frac{4}{6} + 4A + 4B + \frac{1}{2} + A\right)$. By comparing with the conditions $y'(0) = 1$ and $y'(1) = e^4$, we obtain that $4B + A = 1$, and $\frac{4}{6} + 4A + 4B + \frac{1}{2} + A = 1$. It follows that $\frac{4}{6} + 4A + \frac{1}{2} = 0$ and $A = -\frac{7}{24}$, $B = \frac{31}{96}$.

So the solution of the (1.1) with the given conditions is

$$y(x) = e^{4x}\left(\frac{x^3}{6} - \frac{7x}{24} + \frac{31}{96}\right). \tag{1.12}$$

□

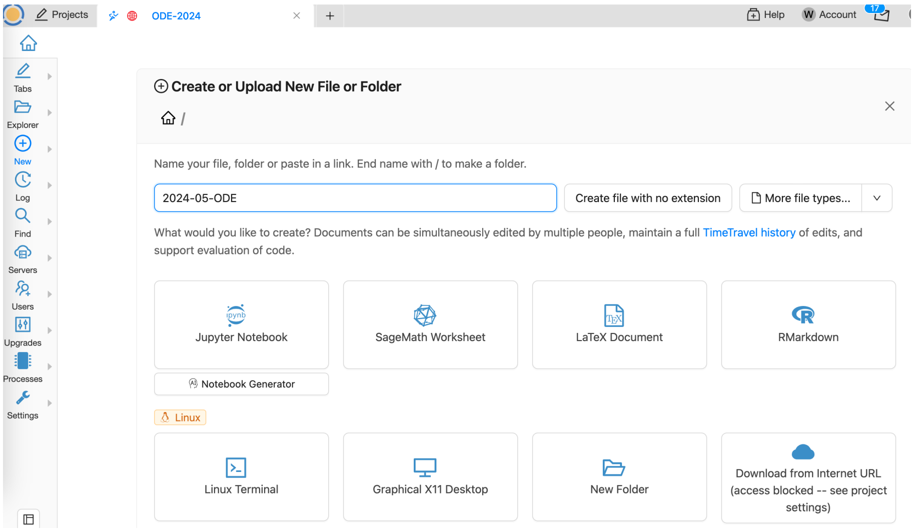
Remark 1.1. One can also use the Laplace transform ([3, Ch. 20]) to solve the (1.1). The transformation (1.6) is simple from Laplace transform point-view.

2 SAGEMATH: AN APPLICATION

SageMath [4] is a free open-source mathematics software system. The homepage is `https://www.sagemath.org/`. One can install it or use it online. For simplicity, we use the cloud version Sage on CoCalc.

Step 1. Visit `https://cocalc.com/features/sage` and sign up for a free CoCalc account.

Step 2. After signing in, create a project `https://cocalc.com/projects`. We call the project **ODE-2024**. Then we can create a SageMath Worksheet.



Step 3. In the SageMath Worksheet, we write the following code.

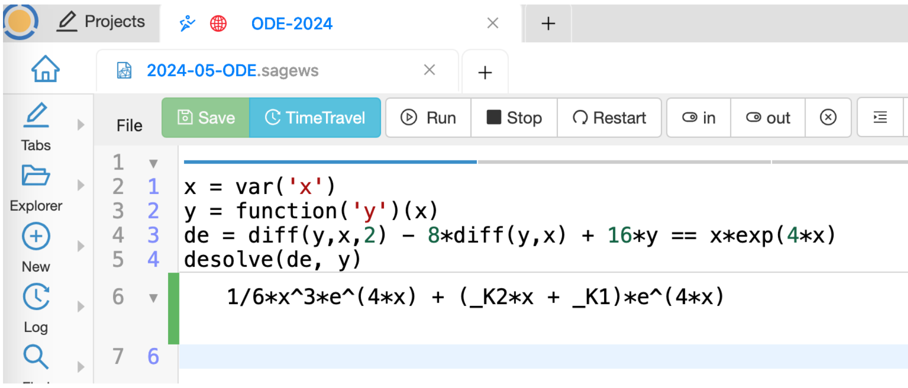
```
x = var('x')
y = function('y')(x)
de = diff(y,x,2) - 8*diff(y,x) + 16*y == x*exp(4*x)
desolve(de, y)
```

The first line of the code defines the variable x. The second line defines y as a function of x. The third line defines the ODE, which is (1.1). The forth line solves the ODE by using the built-in function `desolve()`. More information of solving ordinary differential equations can be found at [5].

Step 4. We can run the code now by clicking button **Run**. It gives the solution of the general form, which is precisely (1.10):

```
1/6*x^3*e^(4*x) + (_K2*x + _K1)*e^(4*x)
```

So the SageMath solves the ODE and finds the general solution.



3 GENERAL ARTIFICIAL INTELLIGENCE.

The ODE serves as a good example to demonstrate that General AI cannot solve certain mathematical problems or, at the very least, have significant limitations. We have tested several General AI systems, including ChatGPT and Copilot.

3.1 Two tests on ChatGPT 3.5, incorrect solutions

We tested this ODE on ChatGPT version 3.5 on May 16, 2024, with the prompt that

```
Solve the ODE y''(x)-8*y'(x)+16*y(x)=x*exp^{4x}, where y is a
funtion of variable x.
```

The response to the prompt is available at [6]. We also attach ChatGPT's response on the last page. It is a useful exercise to verify that ChatGPT's result is incorrect, as stated on the last page:

ChatGPT can make mistakes. Check important info.

One reason is that General AI is based on large language models and statistics, rather than on mathematical logic.

There are two type of errors.

The *first error* is the assumption of the form of particular solution $y_p(x) = Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}$, as explained in (1.5).

The *second error* occurs in the computation of $y_p'(x)$. Specifically, it omits the term related to C in the expression of $y_p'(x)$. Consequently, the computations of both $y_p'(x)$ and $y_p''(x)$ are completely incorrect.

We do not understand the mechanism why ChatGPT makes the second error. If it did not make this mistake, it would have the opportunity to recognize that there is no solution to this calculation. This would lead it back to the first error, allowing it to discover that the initial assumption is incorrect. Additionally, it could test more hypotheses to find the correct solution.

Another version of ChatGPT 3.5's response to the same prompt on June 5, 2024, is available at [7]. It is also incorrect for similar reasons. The *first error* is the assumption form of particular solution. The *second error* occurs in the computation of $y_p''(x)$.

3.2 A test on ChatGPT 3.5 with improved prompt, correct solutions

We improved the prompt as follows in the ChatGPT 3.5, and tested it on June 5, 2024.

```
Solve the ODE y''(x)-8*y'(x)+16*y(x)=x*exp^{4x}, where y is a
function of variable x. Find a particular solution of the form
y(x)=K*(x^3)*exp^{4x}.
```

It generates the correct solutions. The full response is available at [8]. The improved prompt provides the exact information needed for the correct guess form for the particular solution of the ODE, which can be derived through human mathematical logic as shown in (1.6). We do not know if the current general AI can solve this ODE directly without this additional information.

4 CONCLUSION.

This ODE example highlights the rigor and beauty of mathematics, demonstrates the power of SageMath, and reveals the current limitations of general artificial intelligence in solving complex mathematical problems. The quality of ChatGPT responses is entirely dependent on the prompts that provided.

REFERENCES

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[4] 'SageMath Mathematical Software System - Sage'. https://www.sagemath.org/.

[5] The Sage Development Team, 'Solving ordinary differential equations - Symbolic Calculus'. https://doc.sagemath.org/html/en/reference/calculus/sage/calculus/desolvers.html#sage.calculus.desolvers.desolve.

[6] 'ChatGPT 3.5 gives the incorrect solutions of an ODE on 20240516'. https://chatgpt.com/share/dfe5d47f-2f40-49ab-b9ee-958ddff1073b.

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ChatGPT 3.5

Solve the ODE y"(x)-8*y'(x)+16*y(x)=x*exp^{4x}, where y is a function of variable x.

To solve the given ordinary differential equation (ODE), we'll first find the complementary solution, then the particular solution.

The characteristic equation corresponding to the homogeneous part of the ODE is:
$r^2 - 8r + 16 = 0$.

Solving this quadratic equation, we get a repeated root $r = 4$. Therefore, the complementary solution is:
$y_c(x) = (c_1 + c_2 x)e^{4x}$.

Now, for the particular solution, we can use the method of undetermined coefficients. We assume the particular solution has the form $y_p(x) = Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}$.

Differentiating $y_p(x)$ twice and substituting into the original ODE, we find the values of A, B, and C that satisfy the equation.

Let's proceed with finding A, B, and C:
$y_p(x) = Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}$
$y_p'(x) = (2Ax + B)e^{4x} + 4Ax^2e^{4x} + Be^{4x}$
$y_p''(x) = (2A + 8Ax + 4B)e^{4x} + 8Axe^{4x}$

Substituting these into the original ODE:

$(2A + 8Ax + 4B)e^{4x} + 8Axe^{4x} - 8((2Ax + B)e^{4x} + 4Ax^2e^{4x} + Be^{4x}) + 16(Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}) = xe^{4x}$

Grouping like terms:

$(2A + 8Ax + 4B - 8B)e^{4x} + (8Ax - 8Ax)e^{4x} + (16Ax^2 - 32Ax^2 + 16Bx)e^{4x} + (16C)e^{4x} = xe^{4x}$

Simplify and equate coefficients:

$(-4B)e^{4x} + (16Bx)e^{4x} + (16C)e^{4x} = xe^{4x}$

Comparing coefficients:

$-4B = 0 \implies B = 0$
$16B = 1 \implies B = \frac{1}{16}$
$16C = 0 \implies C = 0$

So, we have $B = \frac{1}{16}$, and A and C are arbitrary constants.

Thus, the particular solution is:
$y_p(x) = \frac{1}{16}xe^{4x}$.

Now, the general solution is the sum of the complementary and particular solutions:
$y(x) = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{4x} + \frac{1}{16}xe^{4x}$.

This is the general solution to the given ODE.

Message ChatGPT