

From 500 to a General Formula: Exploring a Floor Identity

WANMIN LIU
wanminliu@gmail.com

12th October 2026

1 THE PROBLEM AND A CHANGE OF PERSPECTIVE

The original problem. Prove that there exists a real number w such that

$$\sum_{k=0}^{500} \left\lfloor \frac{x + 2^k}{2^{k+1}} \right\rfloor = \lfloor x \rfloor - \lfloor wx \rfloor \quad \text{for every } x \in \mathbb{R}. \quad (1)$$

Before trying to solve the problem, let us test a special value of x . Taking $x = 2^{501}$, we have

$$\left\lfloor \frac{2^{501} + 2^k}{2^{k+1}} \right\rfloor = 2^{500-k}, \quad 0 \leq k \leq 500.$$

Therefore,

$$\sum_{k=0}^{500} \left\lfloor \frac{2^{501} + 2^k}{2^{k+1}} \right\rfloor = \sum_{k=0}^{500} 2^{500-k} = 2^{501} - 1.$$

If the identity holds for every real x , we must have

$$2^{501} - 1 = 2^{501} - \lfloor 2^{501}w \rfloor,$$

and hence

$$\lfloor 2^{501}w \rfloor = 1.$$

By the definition of the floor function,

$$1 \leq 2^{501}w < 2.$$

Dividing by the positive number 2^{501} gives

$$2^{-501} \leq w < 2^{-500}.$$

This gives an interval for w , but can we narrow it down? Let us generalize our choice of x by taking

$$x = 2^{500+s}, \quad s \in \mathbb{Z}_{>0}.$$

For every $0 \leq k \leq 500$, we have

$$\left\lfloor \frac{2^{500+s} + 2^k}{2^{k+1}} \right\rfloor = 2^{499+s-k}.$$

Consequently,

$$\sum_{k=0}^{500} \left\lfloor \frac{2^{500+s} + 2^k}{2^{k+1}} \right\rfloor = \sum_{k=0}^{500} 2^{499+s-k} = 2^{500+s} - 2^{s-1}.$$

If the original identity holds for every real x , then

$$\left\lfloor 2^{500+s} w \right\rfloor = 2^{s-1}.$$

By the definition of the floor function,

$$2^{s-1} \leq 2^{500+s} w < 2^{s-1} + 1.$$

Dividing by the positive number 2^{500+s} , we obtain

$$2^{-501} \leq w < 2^{-501} + 2^{-(500+s)}.$$

For $s = 1$, this is precisely the interval obtained from $x = 2^{501}$. As s increases, the upper bound approaches the lower bound 2^{-501} . Since the inequalities hold for every positive integer s , we conclude that

$$w = 2^{-501}.$$

We have found the only possible value of w , but we have not yet proved that it works for every real x . Our choices of special values helped us determine w ; they gave us no clear method for proving the identity for an arbitrary real x . How can we proceed?

There seems to be nothing mathematically special about 500: the summands follow the same pattern regardless of where we stop. Perhaps making the problem more general will help us find a proof. Let us therefore replace 500 by a nonnegative integer m . Since the unknown number may depend on m , we denote it by w_m .

We are thus led to a more general problem: for each $m \in \mathbb{Z}_{\geq 0}$, find a real number w_m such that

$$\sum_{k=0}^m \left\lfloor \frac{x + 2^k}{2^{k+1}} \right\rfloor = \lfloor x \rfloor - \lfloor w_m x \rfloor \quad \text{for every } x \in \mathbb{R}. \quad (2)$$

Our purpose in introducing m is not merely to find another unknown coefficient: we already know the necessary value when $m = 500$. We want to discover *how to prove* the identity. At first sight, the generalized problem may seem more difficult, but it allows us to begin with $m = 0$ and $m = 1$. We will still choose special values of x to find candidate coefficients. The more important question is how to verify the equalities for every real x , and whether those proofs reveal a general structure.

2 EXPLORING THE GENERALIZED PROBLEM

2.1 The case $m = 0$

Equation (2) becomes

$$\left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor - \lfloor w_0 x \rfloor. \quad (3)$$

Step 1: Find a candidate.

We use the same strategy as in the original problem: choose a family of special values of x , obtain inequalities, and let the parameter tend to infinity. This time, take $x = 2n - 1$ with $n \in \mathbb{Z}$. We choose $n \geq 1$, so that $2n - 1 > 0$. Then

$$\left\lfloor \frac{(2n-1)+1}{2} \right\rfloor = n, \quad \lfloor 2n-1 \rfloor = 2n-1.$$

If (3) holds for all x , it follows that

$$n = 2n - 1 - \lfloor w_0(2n-1) \rfloor,$$

and hence

$$\lfloor w_0(2n-1) \rfloor = n - 1.$$

By the definition of the floor function,

$$n - 1 \leq w_0(2n-1) < n.$$

Our choice $n \geq 1$ ensures that $2n - 1 > 0$. Dividing by this positive number gives

$$\frac{n-1}{2n-1} \leq w_0 < \frac{n}{2n-1}.$$

Since the assumed identity holds for every real x , these inequalities hold for every positive integer n . Both bounds tend to $\frac{1}{2}$ as $n \rightarrow \infty$. Therefore,

$$w_0 = \frac{1}{2}.$$

This is a *necessary condition*. So far, we have not proved that this value works for every real x .

Step 2: Verify the candidate.

We recall the following special case of *Hermite's floor identity*:

$$\lfloor y \rfloor + \left\lfloor y + \frac{1}{2} \right\rfloor = \lfloor 2y \rfloor, \quad y \in \mathbb{R}. \quad (4)$$

A proof is given in the appendix.

Taking $y = x/2$, we obtain

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor.$$

Hence (3) holds for every real x with $w_0 = \frac{1}{2}$.

2.2 The case $m = 1$

Now the desired identity becomes

$$\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{4} \right\rfloor = \lfloor x \rfloor - \lfloor w_1 x \rfloor. \quad (5)$$

We follow the same two-step strategy.

Step 1: Find a candidate.

As in the first case, take $x = 2n - 1$ with $n \in \mathbb{Z}$.

To simplify the second floor term as well, choose $n = 2t$ with $t \in \mathbb{Z}_{>0}$. Thus our test values are

$$x = 4t - 1 > 0.$$

Then

$$\left\lfloor \frac{x+1}{2} \right\rfloor = 2t, \quad \left\lfloor \frac{x+2}{4} \right\rfloor = \left\lfloor t + \frac{1}{4} \right\rfloor = t.$$

Thus (5) implies

$$3t = 4t - 1 - \lfloor (4t - 1)w_1 \rfloor,$$

or equivalently,

$$\lfloor (4t - 1)w_1 \rfloor = t - 1.$$

By the definition of the floor function,

$$t - 1 \leq (4t - 1)w_1 < t.$$

Our choice $t \geq 1$ ensures $4t - 1 > 0$. Dividing by this positive number gives

$$\frac{t-1}{4t-1} \leq w_1 < \frac{t}{4t-1}.$$

Since the assumed identity holds for every real x , these inequalities hold for every $t \in \mathbb{Z}_{>0}$.

Letting $t \rightarrow \infty$, we obtain the necessary value

$$w_1 = \frac{1}{4}.$$

Step 2: Verify the candidate.

Apply (4) twice. With $y = x/4$,

$$\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+2}{4} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor. \quad (6)$$

With $y = x/2$,

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor. \quad (7)$$

Adding these equalities and cancelling the common term $\lfloor x/2 \rfloor$ gives

$$\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+2}{4} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor.$$

This proves (5) for every $x \in \mathbb{R}$ with $w_1 = \frac{1}{4}$.

What have we discovered?

The first two cases give

$$w_0 = \frac{1}{2}, \quad w_1 = \frac{1}{4},$$

suggesting that

$$w_m = \frac{1}{2^{m+1}}$$

might work in general.

More importantly, the proof for $m = 1$ reveals a structure: applying Hermite's identity at successive scales allows intermediate terms to cancel. This suggests a way to prove the conjecture for arbitrary m .

2.3 The general case

We now prove that the proposed formula works for every nonnegative integer m .

Theorem 2.1. *For every $m \in \mathbb{Z}_{\geq 0}$ and every $x \in \mathbb{R}$,*

$$\sum_{k=0}^m \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor + \left\lfloor \frac{x}{2^{m+1}} \right\rfloor = \lfloor x \rfloor. \quad (8)$$

In particular, $w_m = 2^{-(m+1)}$ satisfies (2).

Proof. The key is again Hermite's identity

$$\lfloor y \rfloor + \left\lfloor y + \frac{1}{2} \right\rfloor = \lfloor 2y \rfloor.$$

For each $j = 1, \dots, m+1$, take $y = x/2^j$. Then

$$\left\lfloor \frac{x}{2^j} \right\rfloor + \left\lfloor \frac{x+2^{j-1}}{2^j} \right\rfloor = \left\lfloor \frac{x}{2^{j-1}} \right\rfloor.$$

When $m = 0$, this is just one application of Hermite's identity. For larger m , the successive equalities have the following form:

$$\begin{aligned} \left\lfloor \frac{x}{2^{m+1}} \right\rfloor + \left\lfloor \frac{x+2^m}{2^{m+1}} \right\rfloor &= \left\lfloor \frac{x}{2^m} \right\rfloor, \\ \left\lfloor \frac{x}{2^m} \right\rfloor + \left\lfloor \frac{x+2^{m-1}}{2^m} \right\rfloor &= \left\lfloor \frac{x}{2^{m-1}} \right\rfloor, \\ &\vdots \\ \left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+2}{4} \right\rfloor &= \left\lfloor \frac{x}{2} \right\rfloor, \\ \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor &= \lfloor x \rfloor. \end{aligned}$$

Add all these equations. Every intermediate term $\lfloor x/2^j \rfloor$, $1 \leq j \leq m$, appears once on each side and cancels. What remains is

$$\left\lfloor \frac{x}{2^{m+1}} \right\rfloor + \left\lfloor \frac{x+2^m}{2^{m+1}} \right\rfloor + \left\lfloor \frac{x+2^{m-1}}{2^m} \right\rfloor + \dots + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor,$$

which is exactly (8). □

Returning to the original problem, take $m = 500$. The identity holds for every $x \in \mathbb{R}$ with

$$w = \frac{1}{2^{501}}.$$

Together with the necessary condition established in Section 1, this shows that $w = 2^{-501}$ is the *unique* real number satisfying (1).

3 REFLECTION: FROM A SPECIFIC PROBLEM TO A GENERAL PROOF

Our exploration involved two different generalizations. First, with the upper limit fixed at 500, we replaced the single test value $x = 2^{501}$ by the family $x = 2^{500+s}$. The resulting intervals narrowed as s increased. Taking a limit determined the only possible value of w , but did not tell us how to prove the identity for arbitrary real x .

This difficulty motivated our second generalization: we replaced 500 by a variable m . The aim was no longer simply to find the coefficient, but to discover a method of proof. The general problem allowed us to examine $m = 0$ and $m = 1$. In each case we again chose a family of special values of x and used a limit to find a candidate. The method for finding necessary conditions was familiar; what was new was the proof of sufficiency using Hermite's identity.

The case $m = 1$ was particularly revealing. Its proof showed that applying Hermite's identity at successive scales allows intermediate terms to cancel. This provided a structure that could be repeated for arbitrary m .

Testing special values can help us discover what a coefficient must be, but does not establish an identity for every real number. The first generalization helped us determine the necessary value; the second helped us understand why the proposed value works.

Sometimes, making a problem more general is precisely what makes it easier to prove.

A A SPECIAL CASE OF HERMITE'S FLOOR IDENTITY

We prove that, for every $y \in \mathbb{R}$,

$$\lfloor y \rfloor + \left\lfloor y + \frac{1}{2} \right\rfloor = \lfloor 2y \rfloor.$$

Proof. Let $n = \lfloor y \rfloor$. Then $y = n + r$ for some $0 \leq r < 1$. Hence

$$\lfloor y \rfloor + \left\lfloor y + \frac{1}{2} \right\rfloor = 2n + \left\lfloor r + \frac{1}{2} \right\rfloor, \quad \lfloor 2y \rfloor = 2n + \lfloor 2r \rfloor.$$

It remains to show that $\left\lfloor r + \frac{1}{2} \right\rfloor = \lfloor 2r \rfloor$.

If $0 \leq r < \frac{1}{2}$, both sides equal 0. If $\frac{1}{2} \leq r < 1$, both sides equal 1. Thus the identity holds for every $y \in \mathbb{R}$. $\square$